Bootstrap percolation and interacting particle systems with kinetic constraints

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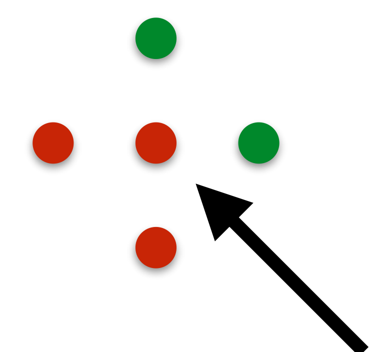
(a joint project with L. Mareche, R. Morris and C. Toninelli)

ERC grant “Malig”

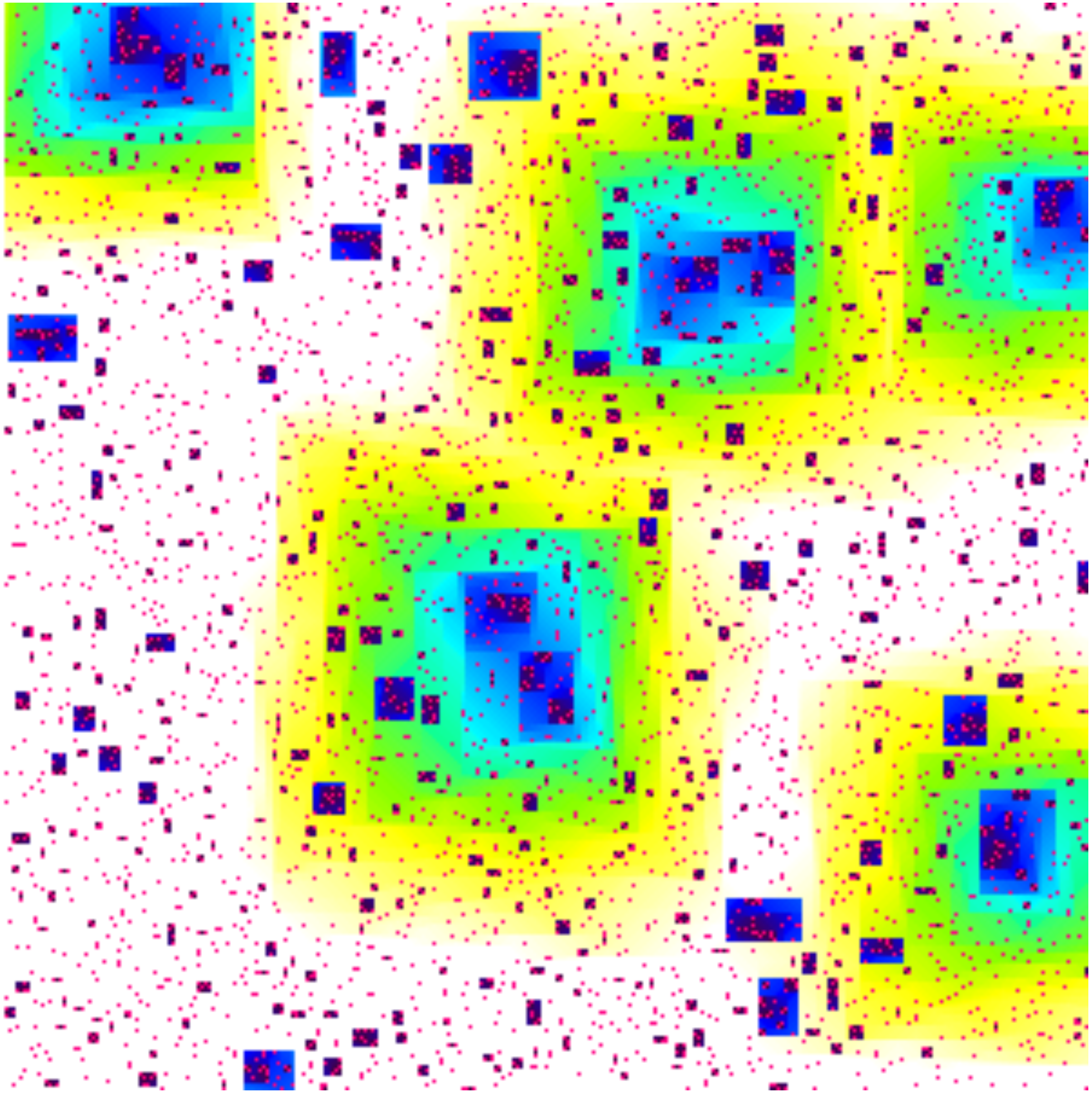
Bootstrap percolation: a first example

- At time t=0 vertices of $\mathbb{Z}^2$ are i.i.d. infected with prob p and healthy with prob 1-p.
- At time t=1 healthy vertices with at least 2 infected neighbours become infected.



- Iterate…

The bootstrap process is monotone (infection never heals)



2-neighbor process

(A. Holroyd)

Colors measure # iterations of BM necessary to infect a given vertex.

Black < Blu < Azur < Green < Yellow < White

Some basic questions

Q_1: Will the entire lattice become infected eventually ?

A_1: Yes (van Enter '87)

Q_2: How long does it take (w.h.p. or in mean) to infect the origin ?

A_2: Approx Exp($\pi^2/18p$) steps for $p \ll 1$ (A. Holroyd '03)

Q_3: Compute the scaling as $n \to \infty$ of

$$p_c(n) := \inf\{p : \mathrm{Prob}_p(\Lambda_n \text{ becomes eventually infected }) \geq 1/2\}$$

A_3: $p_c(n) = \Theta(1/\log n)$.

A general setting

★ A bootstrap rule $\mathscr{U}$ is a collection $(X_1, X_2, \ldots, X_m)$ of finite subsets of $Z^2 \setminus \{0\}$.

★ $A_t \subseteq Z^2 \;\longleftrightarrow\;$ Set of infected sites at time t.

★ $\mathscr{U}$-bootstrap process:

$$A_{t+1} = A_t \cup \{ x \in Z^2 : \; x + X \subseteq A_t \text{ for some } X \in \mathscr{U} \}$$

$[A]_{\mathscr{U}} := \cup_{t \geq 0} A_t$ the "closure of A" (or final infected set).

Examples

The Duarte model

$\mathscr{U}$ = family of 2-subsets of the South, North and West neighbours of the origin

The "South or West" process (known as the East model 🙂)

$\mathscr{U}$ = family of 1-subsets of the South and North neighbours

The North-East process

$\mathscr{U}$ = North & East neighbours

Critical Probability and Critical Length

- Let Λ_n be the n-torus in Z^2.
- Take the initial infected set $A := A_{t=0}$ p-random.
- Critical probability:

$$p_c(n) = \inf\{\, p: \ \mathrm{Prob}_p(\, [A]_{\mathcal{U}} = \Lambda_n \,) \geq 1/2) \,\}$$

The scaling of $p_c(n)$ as $n \to \infty$ is dictated by the action of the $\mathcal{U}$-process on discrete half-planes (Bollobas, Smith and Uzzell '15)

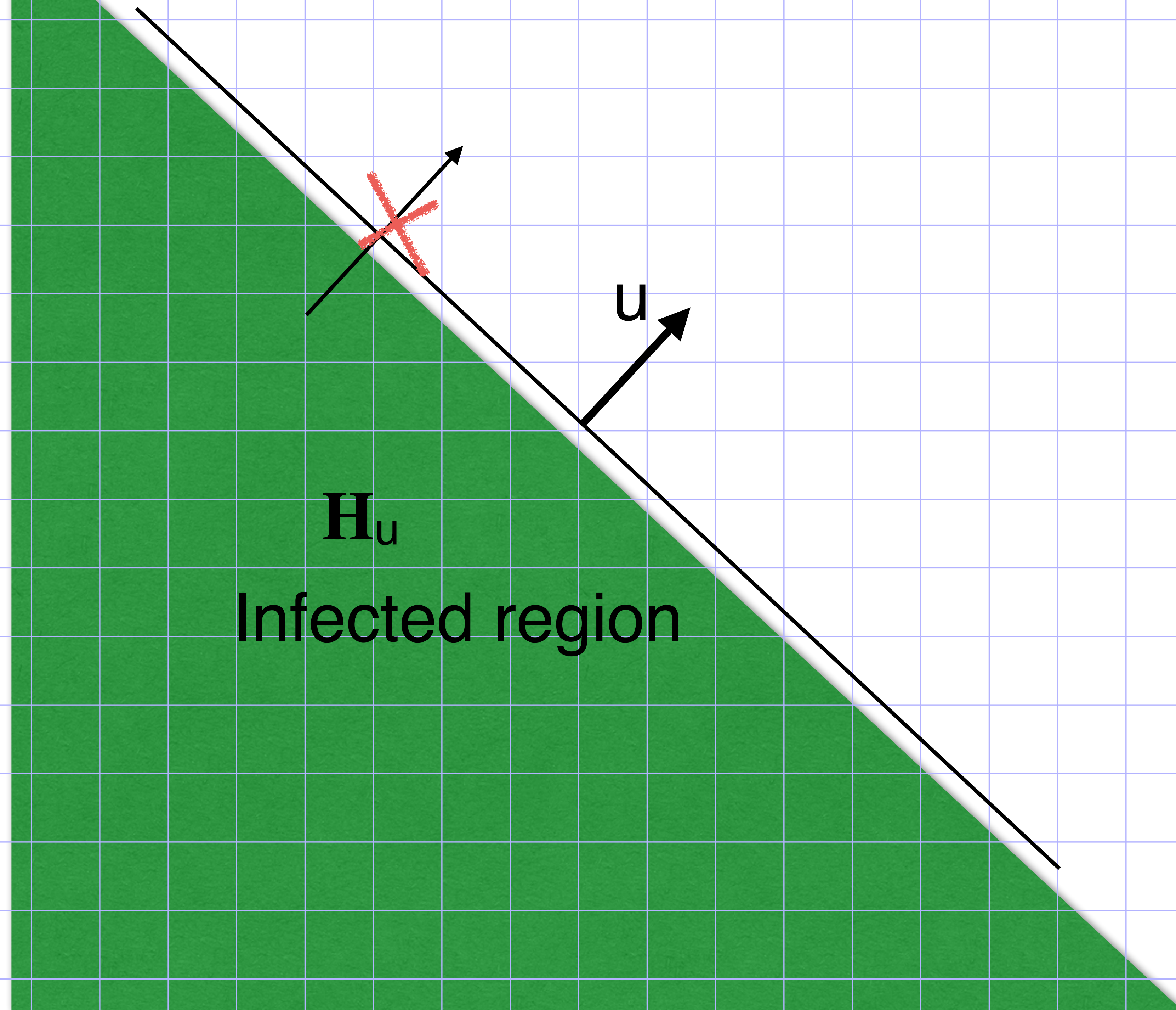
Stable Directions

A direction $u \in S^1$ is ***stable*** if the "$\mathscr{U}$-closure" of the half-plane

$$\mathbf{H}_u = \{x \in Z^2 : \ \langle x,u \rangle < 0 \}$$

is $\mathbf{H}_u$ itself: $[\mathbf{H}_u]_{\mathscr{U}} = \mathbf{H}_u$.

No infection
can cross the line orthogonal to a
stable direction **u**



A general classification

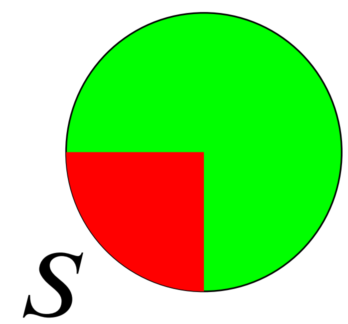
(Bollobas-Smith-Uzzell '15)

Let $\mathcal{S}$ be the set of stable directions. The $\mathcal{U}$-process is:

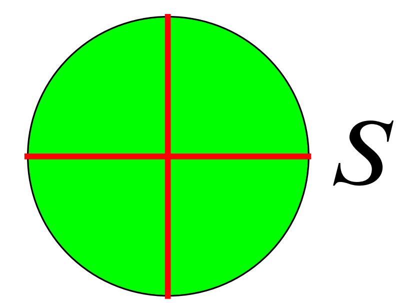
- ***Supercritical*** : $\exists$ open semicircle S with no stable directions.
- ***Critical*** : every semicircle is such that S $\cap$ $\mathcal{S} \neq \emptyset$ and $\exists$ one semicircle with finite intersection with $\mathcal{S}$.
- ***Subcritical*** : every semicircle has ∞ intersection with $\mathcal{S}$.

Examples

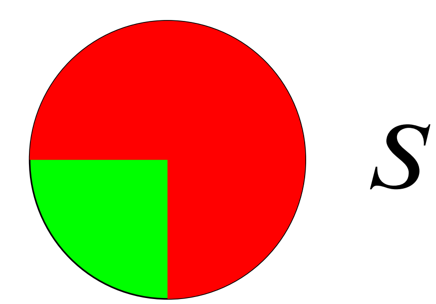
The South-or-West process
(supercritical)



2-neighbour process (critical)



The North-East process (subcritical)



A universality result for $p_c(n)$

(Bollobas-Smith-Uzzell '15, Balister-Bollobas-Przykucki-Smith '16)

Theorem Let $\mathscr{U}$ be an update family.

$\mathscr{U}$ supercritical $\Rightarrow p_c(n) = 1/n^{\Theta(1)}$

$\mathscr{U}$ critical $\Rightarrow p_c(n) = 1/(\log n)^{\Theta(1)}$

$\mathscr{U}$ subcritical $\Rightarrow \liminf p_c(n) > 0.$

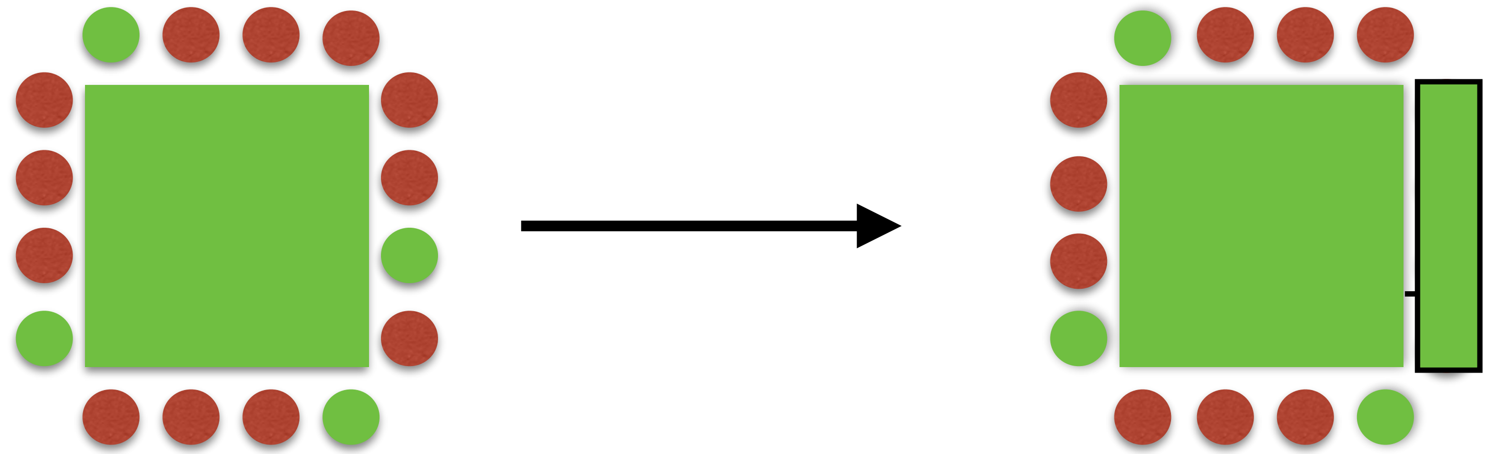
A refinement for critical systems

- In '16 Bollobas, Duminil-Copin, Morris and Smith introduced the notion of "difficulty" α for critical systems and proved $p_c(n) = \Theta(1/(\log n)^{1/\alpha})$.

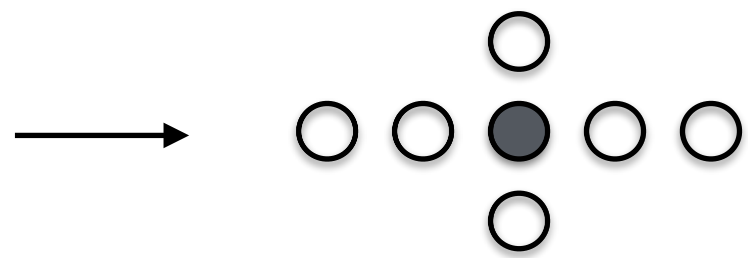
- Roughly, $\mathbf{u} \in S^1$ has difficulty α if, with the help of α extra infected sites, the half-plane $H_{\mathbf{u}}$ is able to grow in the $\mathbf{u}$ direction.

Definition: $\alpha(\mathcal{U}) = \min_S \max_{u \in S} \alpha(u)$

2-neighbor process
$\alpha = 1$



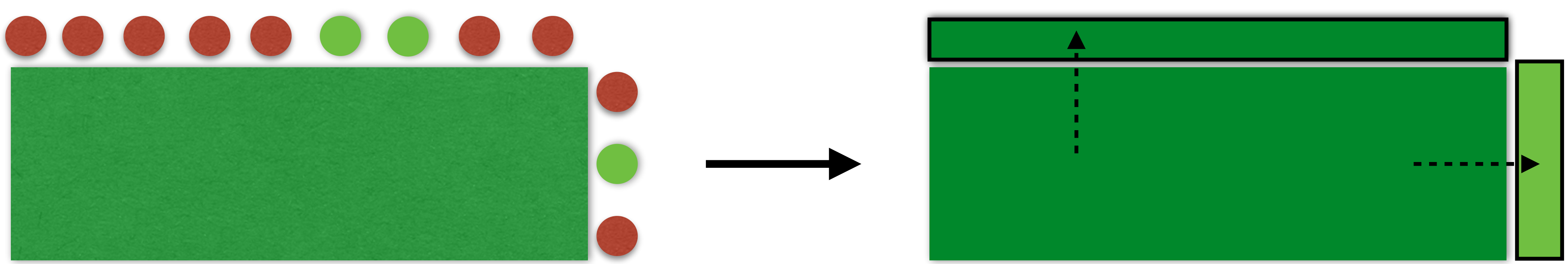
Anisotropic model: at least 3 infected sites among



α_v=2 in the $\updownarrow$ direction

α_h=1 in the $\longleftrightarrow$ direction

$\Rightarrow \ \alpha = \min(\alpha_v, \alpha_h) = 1$



Critical Lengths

Fix $p \ll 1$ and let $L_c = L_c(p)$ be such that

p_c(L_c) = p

- $\mathscr{U}$ supercritical : $L_c(p) = 1/p^{\Theta(1)}$ as $p \searrow 0$.
- $\mathscr{U}$ critical : $L_c(p) = \Theta(\operatorname{Exp}(1/p^{\alpha}))$ as $p \searrow 0$.

Kinetically constrained models

- ⋆ Fix a bootstrap rule $\mathscr{U}$ and $0<p<1$.

The updating rule :

- ✦ Each vertex which could be infected by one step of the $\mathscr{U}$-process becomes infected with rate p and non-infected with rate 1-p independently of the others.

- ✦ All other vertices don't change their state.

KCM: general features

- Reversible interacting particle systems on Z^2.
- Reversible measure μ = product Bernoulli(p)
- Not monotone (infection can heal).
- Not necessarily ergodic (there could be blocked configurations)

Main obstacles for a mathematical analysis

- KCMs are not monotone: more infected vertices in the system may have **unpredictable** consequences.
- Powerful tools like FKG inequalities or Censoring are no longer available.
- Worst case analysis (w.r.t. the initial condition) usually too rough.
- Coercive inequalities (e.g. Poincare' and Log-Sobolev) behave very differently from e.g. the "high temperature Ising model"

Origins of KCMs

(nice review by Evans & Sollich)

G. Fredrickson and H. Andersen in ’84 introduced KCMs as simple models featuring *glassy dynamics* at low temperature :

- *Trivial equilibrium.*
- *Broad spectrum of relaxation time scales.*
- *Heterogenous dynamics.*
- *Possible ergodicity breaking transition.*
- *Aging.*

In physical terms small values of the density ***p*** of the infected (facilitating) sites $\longleftrightarrow$ *small temperature*

The infection Time for the KCM

- $T(\mathscr{U})= \inf \{t \geq 0:$ the origin is infected at time $t\}$

- ***Q1***: How does $T(\mathscr{U})$ grow as $p \searrow 0$ for the stationary process? Does it scale as $L_c(p)$?

- General Bounds :

  (a) W.h.p $T(\mathscr{U}) \geq \varepsilon L_c(p)$ for some small $\varepsilon > 0$ (not sharp !!)

  (b) $E_\mu(T(\mathscr{U})) < C \times (1/p) \times$ "Relaxation Time".

The Relaxation Time $T_{rel}(\mathscr{U})$

Quadratic form of the Markov generator of the KCM:

$$\mathscr{D}(f) = \sum_x \mu(c_x \, Var_x(f))$$

c_x = indicator of x being "infectable" by the BM

local variance=spin flip at x

$$T_{rel} = \inf \{\lambda : Var(f) \leq \lambda \, \mathscr{D}(f) \quad \forall f \}$$

A constrained Poincare' inequality

Bounding the relaxation time

Lower bound: use the variational characterisation and find a good test function F

$$\Rightarrow T_{rel} > \mathrm{Var}(F)/\mathscr{D}(F)$$

Upper bound: find C s.t. $\mathrm{Var}(f) \leq C\,\mathscr{D}(f)$ for all f

$$\Rightarrow T_{rel} \leq C$$

Scaling of the infection time for some models

1-neighbor process

(Cancrini, M. Roberto, Toninelli '08
Pillai & A.Smith '15)

$E_\mu(T(\mathscr{U})) \asymp 1/p^{\Theta(1)}$

explicit

South-or-West process

(Chleboun, Faggionato, M. '15)

$E_\mu(T(\mathscr{U})) \asymp p^{-\log(1/p)(\beta+o(1))}$

$(L_c(p) = 1/p^{\Theta(1)}$!!)

2-neighbor process

(M & Toninelli '16)

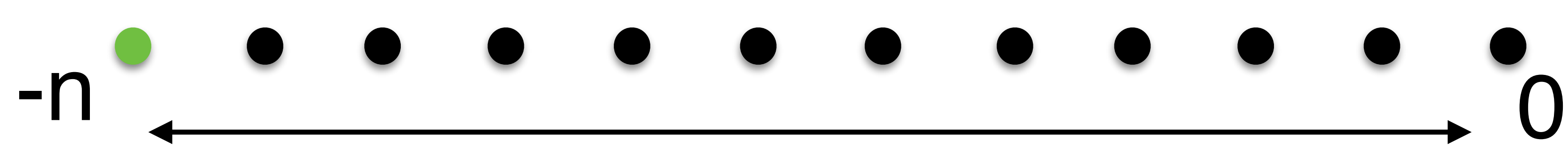
- $E_\mu(T(\mathscr{U})) < (L_c)^{\beta(p)}$

$\beta(p)=O(\log\log L_c)$

On the $p^{-\Theta(\log 1/p)}$ growth for the S-or-W Model.

The system features "Logarithmic energy barriers"

- Consider the East process in 1D (infection goes from left to right BUT not viceversa !)
- Let **Energy := # of infected vertices**
- Assume that the first infected site to the left of x=0 is at **-n**



Theorem [*Chung, Diaconis, Graham* '01] **At least $\log_2(n)$ vertices must be infected before $T(\mathscr{U})$.**

Role of energy barriers

$$P_\mu(T(\mathscr{U}) < t) \leq O(t) \times \mu(\exists \log_2(n) \text{ infected sites})$$

$$\leq O(t) \times p^{\log_2(n)} \times n^{\log_2(n)}$$

Choose $n \sim 1/p^{\varepsilon}$, $\varepsilon \ll 1$,

($1/p$ = equilibrium distance between 2 infected vertices)

$\Downarrow$

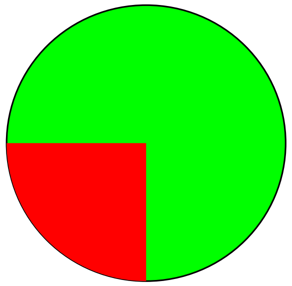
$$P_\mu(T(\mathscr{U}) < t) \leq O(t) \times \exp(-c(\varepsilon)\log(1/p)^2)$$

A refined classification of supercritical systems

Definition. A supercritical $\mathscr{U}$-process is rooted if there exists two non-opposite stable directions. Otherwise it is unrooted .

Examples.

- **Rooted**: the S-or-W process.



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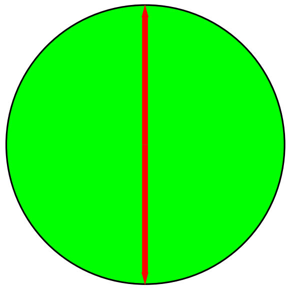
- **Unrooted**: $\mathscr{U}$ ≡ an infected site between the left and right neighbours.



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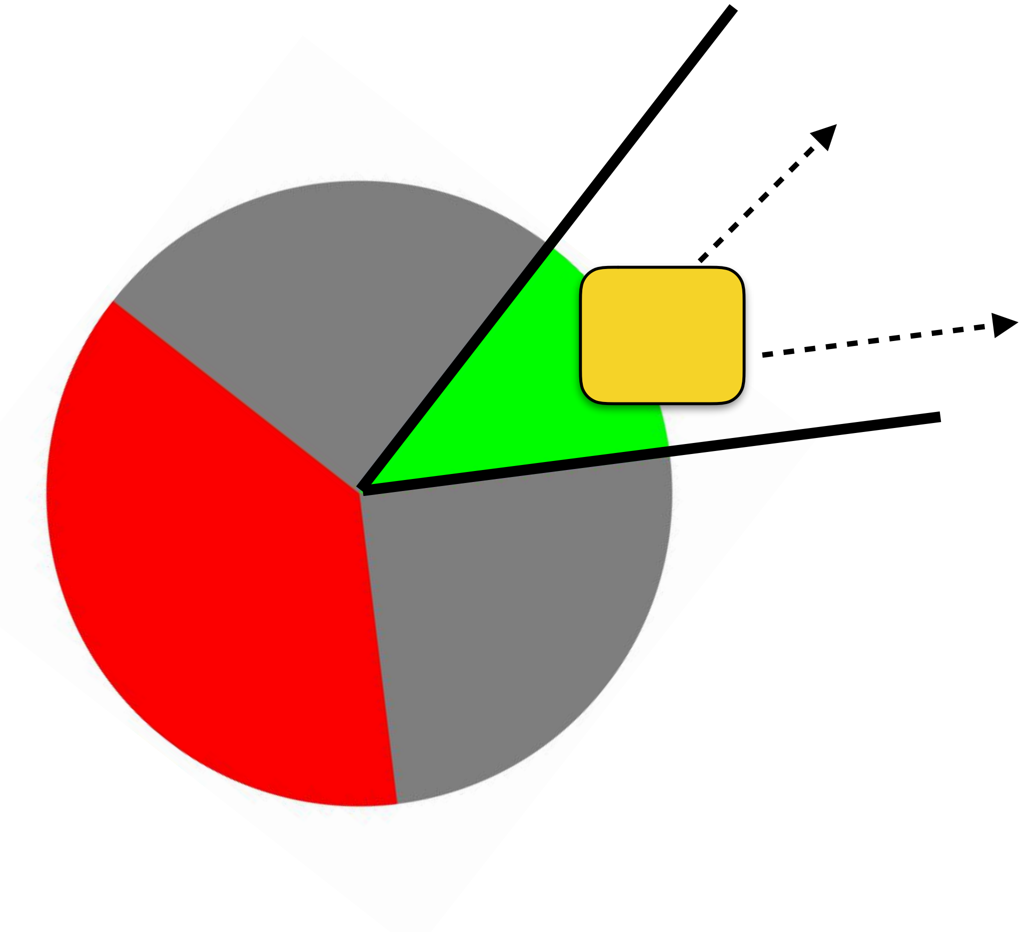
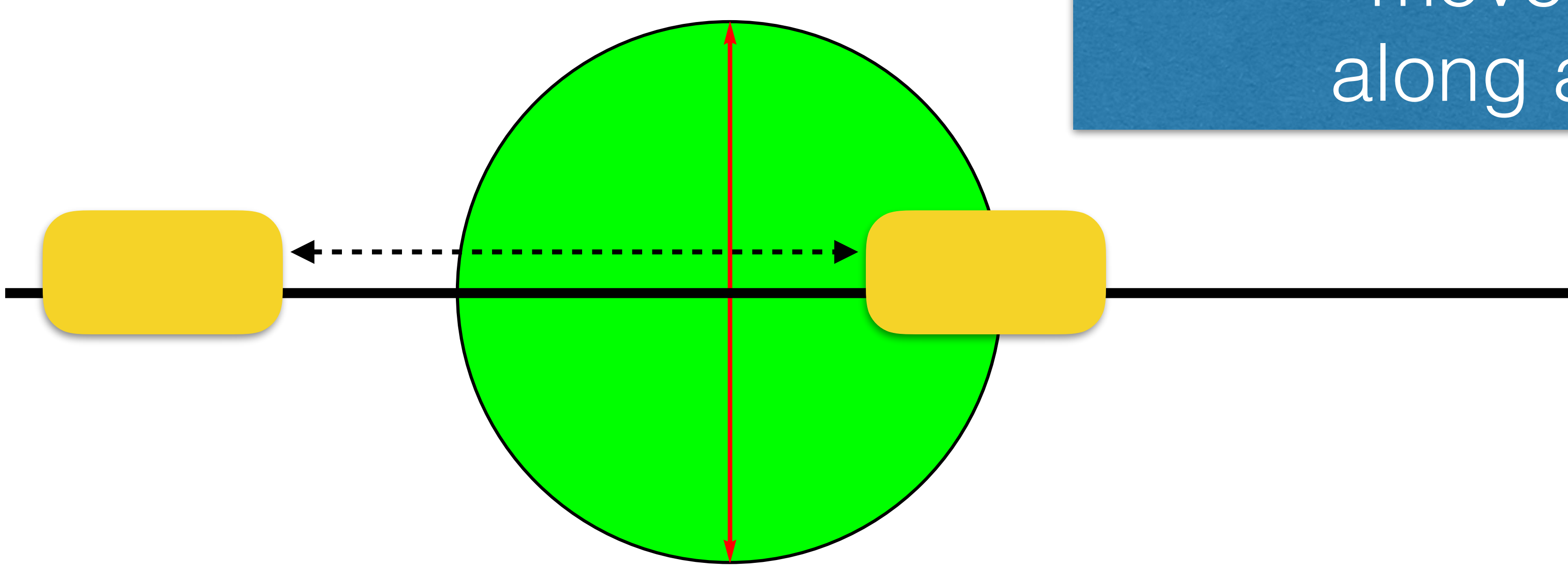
In a *rooted* supercritical system the droplet of infected sites can only move “forward” inside the cone delimited by black lines



In *unrooted* systems a large enough droplet can move back and forth along a given direction



Growth of $T(\mathscr{U})$ for supercritical systems

(with R.Morris, C. Toninelli, L.Mareche)

Let $\mathscr{U}$ be supercritical. As $p \searrow 0$

$T(\mathscr{U}) = p^{-\Theta(1)}$ if $\mathscr{U}$ is unrooted.

$T(\mathscr{U}) = p^{-\Theta(\log(1/p))}$ if $\mathscr{U}$ is rooted.

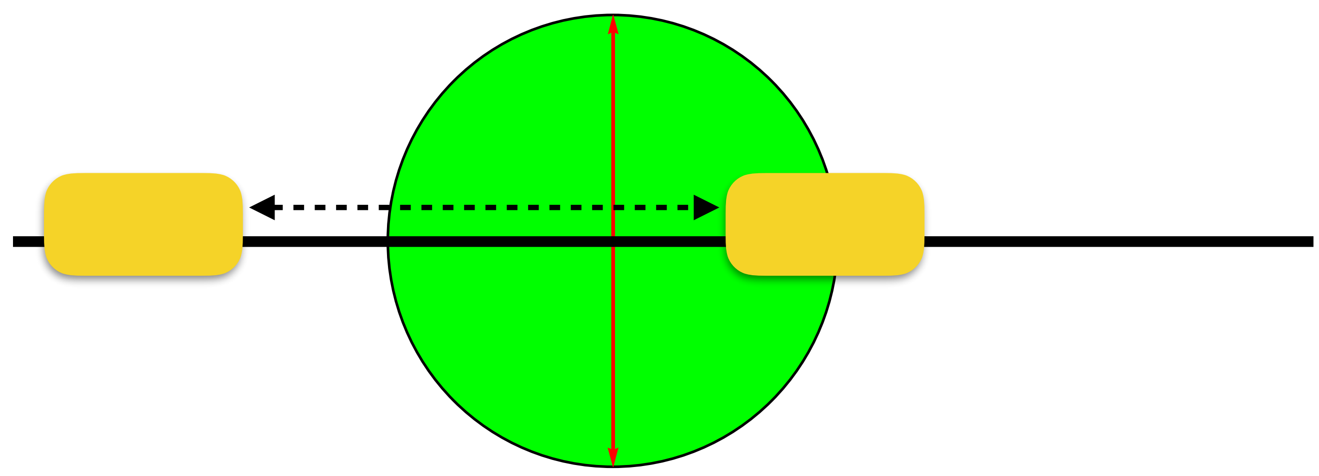
$\downarrow$

Unrooted case: $T(\mathscr{U}) \longleftrightarrow L_c(p)$

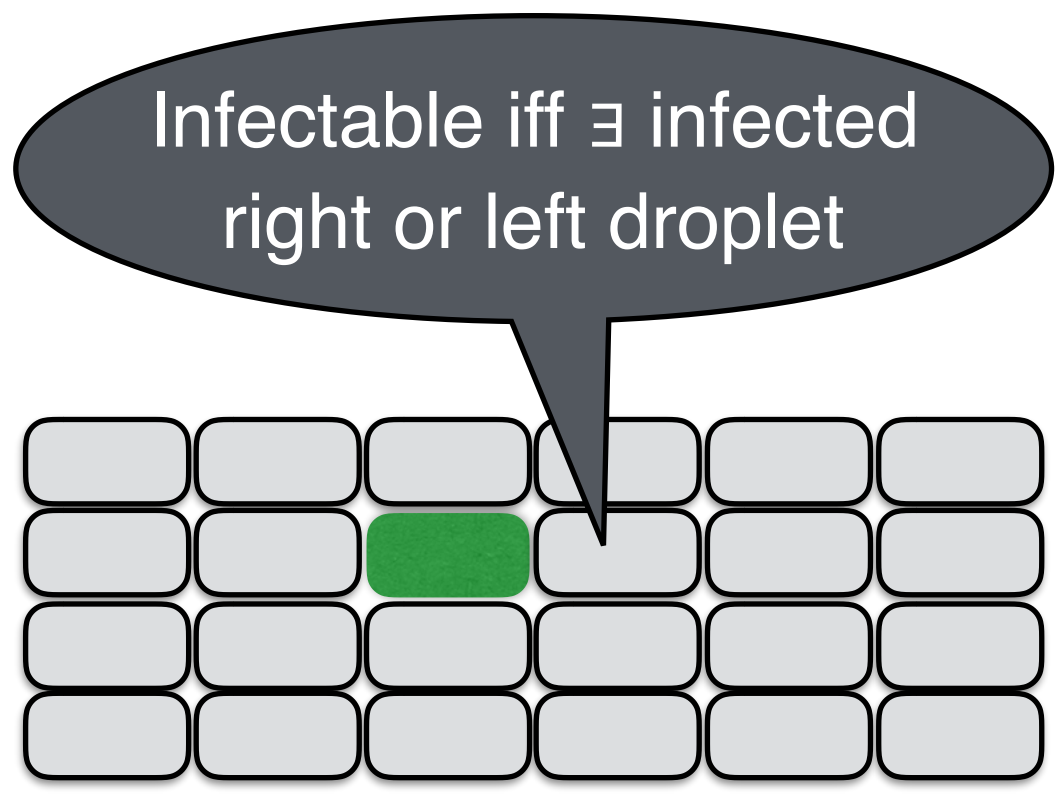
Rooted case: $T(\mathscr{U}) \gg L_c(p)$

The unrooted case

- $\exists$ a direction **u** and a big enough rectangular droplet R parallel to **u** which can shift in both directions **u** and **-u**.



- We can compare the $\mathscr{U}$-KCM to a ***renormalised 1-neighbor*** KCM model in which each vertex of Z^2 actually represents a ***droplet*** *R*.



Renormalised Droplet Model $\Leftrightarrow$ Original Model

In the renormalised model $p_{eff} = p^{Area(droplet\ )}$

For the renormalised droplet model

.

$$T_{rel}(\mathscr{U}_{ren}) \sim 1/\ {p_{eff}}^{\Theta(1)}$$

Comparisons of the Dirichlet forms of the renormalised and original Markov chains $\Rightarrow$

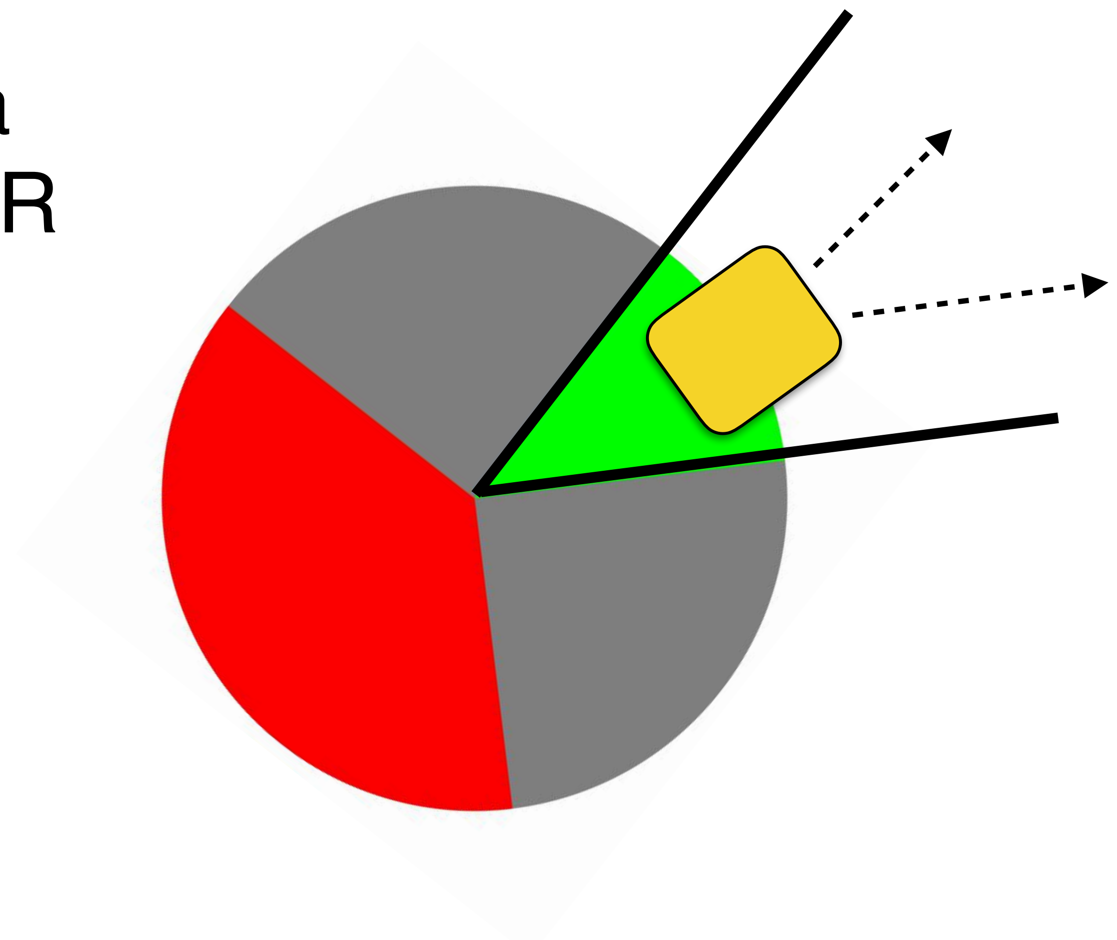
$$T_{rel}(\mathscr{U}) = 1/p^{\Theta(1)}$$

The rooted case

(a) There exists a direction **u** and a big enough rectangular droplet R parallel to **u** which can shift in the **u** direction.

(b) L. Mareche has proved that there exist logarithmic energy barriers.

(c) Again use renormalisation to compare the $\mathscr{U}$-KCM to an effective East model.



Critical systems

Def. A critical system with difficulty α is rooted if there exist two ***non-opposite stable*** directions with **difficulty > $\boldsymbol{\alpha}$.** Otherwise it is unrooted

Theorem [M. Morris, Toninelli]

Let $\mathscr{U}$ be unrooted critical.

$$c_1 L_c \leq E(T(\mathscr{U})) \leq L_c^{\beta}, \qquad \beta = O(\log\log L_c).$$

Conjecture: Let $\mathscr{U}$ be rooted critical.

$$T(\mathscr{U}) = \Theta(\exp(\beta/p^{2\alpha}))$$

Strategy of the proof

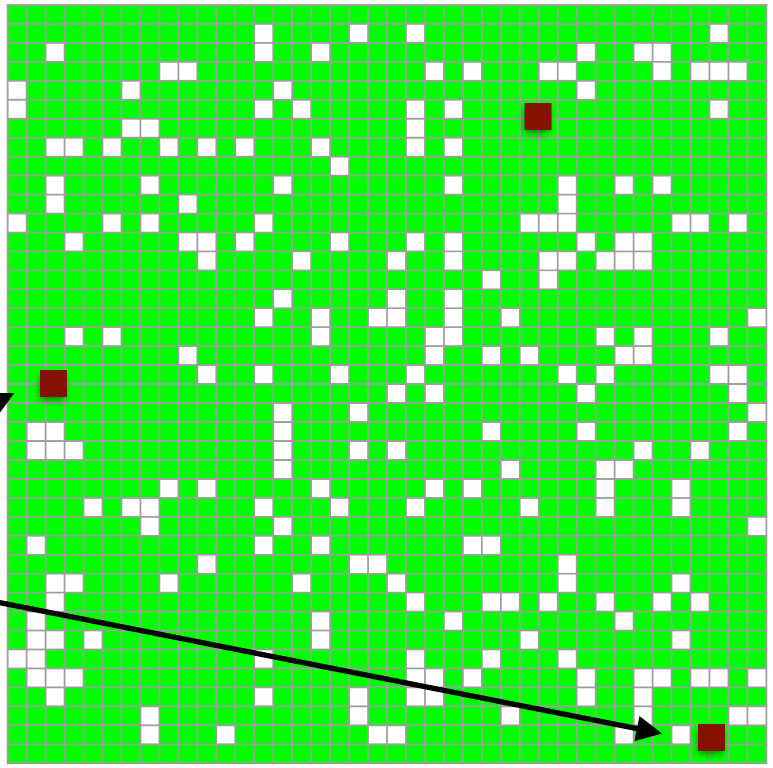
Def: a $L \times L$ box is good if every column/row has an infected site.

If $L \sim Cp^{-1} \times \log(1/p)$, $C > 4$

$\Rightarrow P(\text{Good}) \sim 1$

Good boxes form a (unique) ∞-cluster with holes.

Among the good boxes there are (rare) super-good ones:

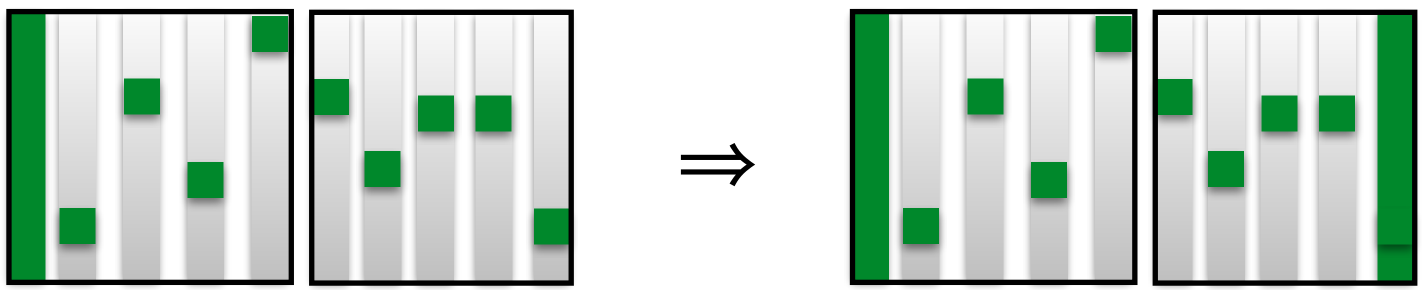


Def: A box is super-good if it is good and 1st row & column completely infected

Prob(SG) ~
Exp(-1/p x polylog(p))

Euristics for $E(T(\mathcal{U})) \leq L_c^\beta$

A SG box can freely travel among good boxes



A. At t=0 w.h.p. a SG box will be found at distance

$$L \sim 1/P(SG)^{1/2}$$

B. In a diffusive time the SG box will move next to the origin.

C. In another time $\propto$ poly(L) the bootstrap map will infect the origin.

Main Obstacles

1. The environment of the good and super-good boxes evolves in time;
2. No monotonicity argument to move infos at $t=0$ to $t > 0$.

Solution:
Bound from above the relaxation time

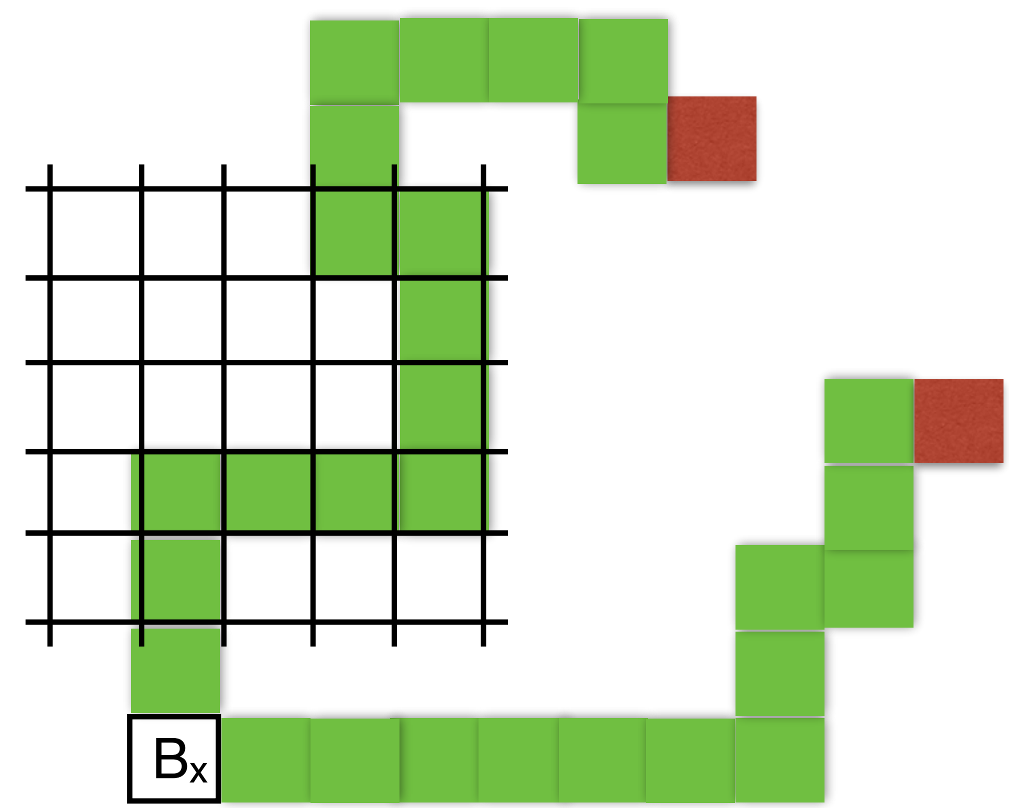
A key constrained Poincare' inequality

$$\mathrm{Var}(f) \leq O(1) \times \sum_{\text{boxes}} \mu\left(c_x \mathrm{Var}_{B_x}(f)\right)$$

indicator these paths exist

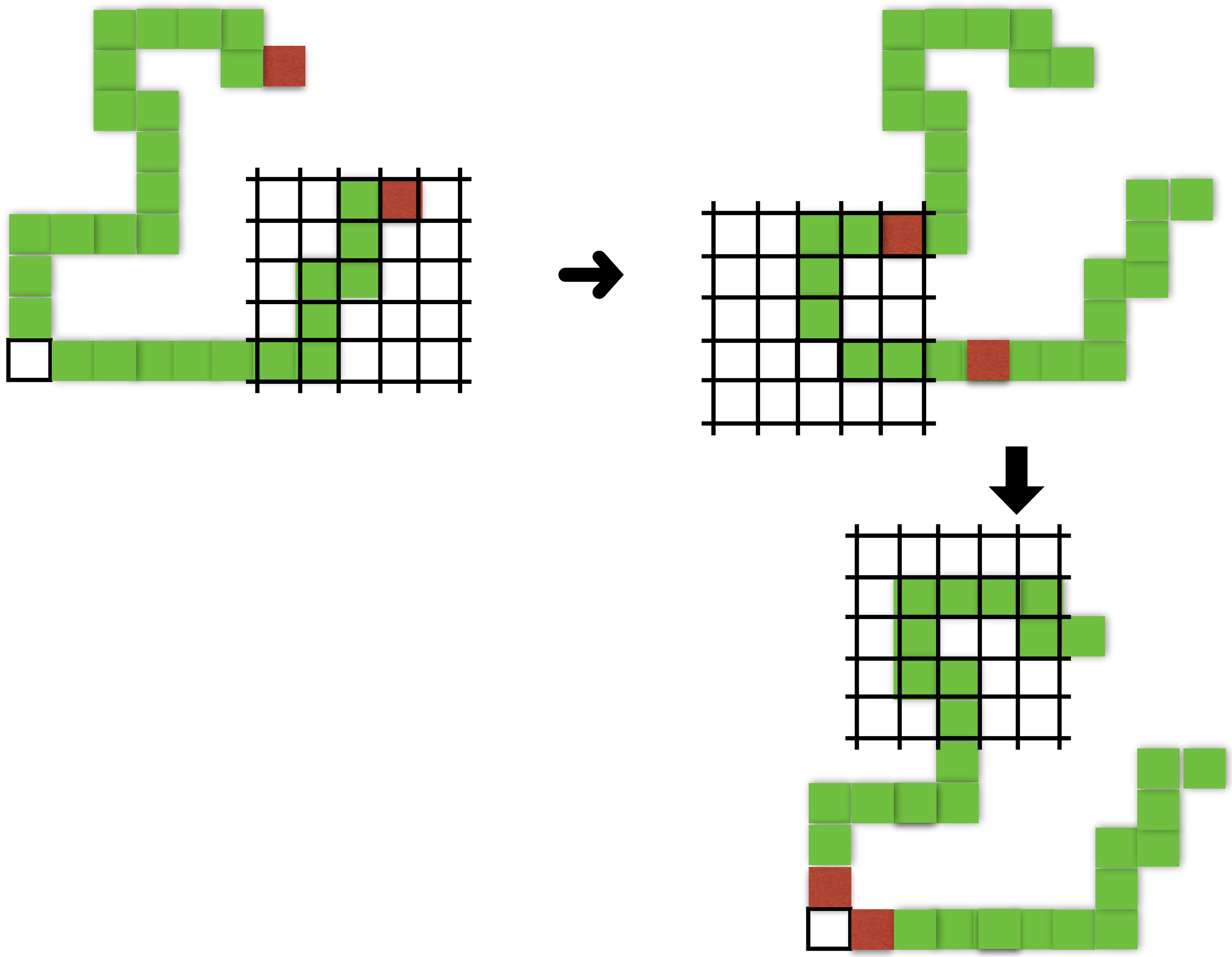
$\Longleftrightarrow$ Good box

$\Longleftrightarrow$ Super-Good box



Max length of the paths =

O(Exp(1/p x polylog))

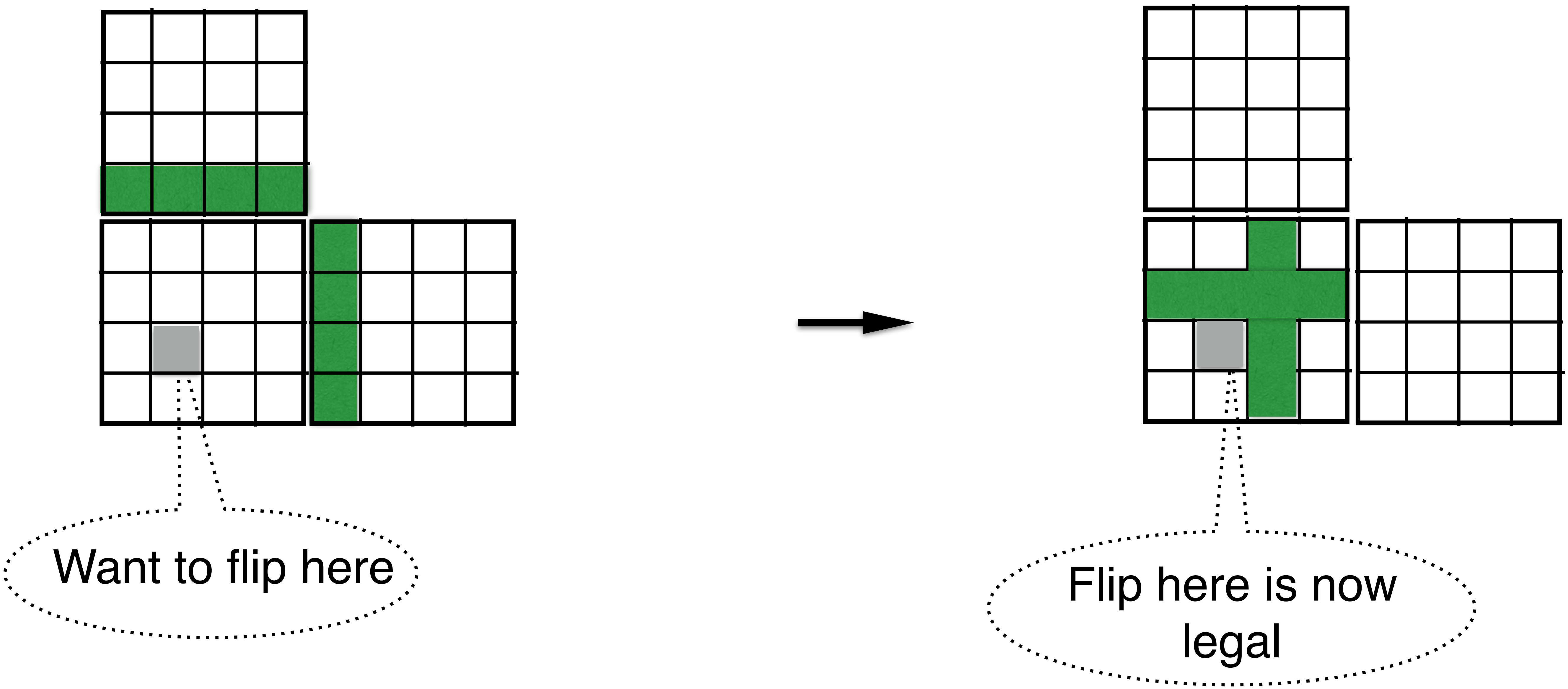


Using canonical paths techniques for reversible Markov chains

the SG boxes are there

$$\mathrm{Var}(f) \le \widetilde{\exp}(1/p) \sum_{\text{boxes}} \mu\left(\lambda_x \mathrm{Var}_{B_x}(f)\right)$$

Conclusion



$$\Rightarrow \ \mathrm{Var}(f) \le \widetilde{\exp}(1/p) \sum_{\text{boxes}} \mu\left(c_x \mathrm{Var}_{B_x}(f)\right)$$

the true constraints

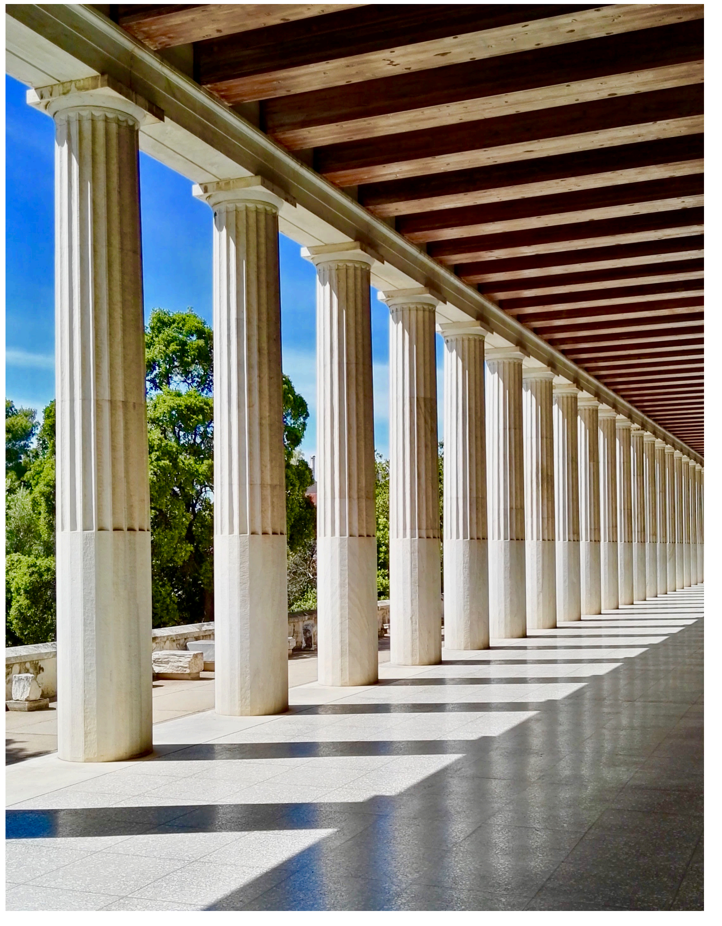
Summary

- Characteristic **time scales** for KCM may scale very differently from the characteristic **length scales** of the bootstrap percolation model.

The new feature of KCM is the possible presence of "**energy barriers**" that the dynamics has to overcome.

A more **refined classification** seems to be able to characterise systems for which **energy barriers matter**.

A **universality** result on the scaling of the infection time emerges.



Thank you