Uniform Price Auctions: Equilibria and Efficiency*

Evangelos Markakis[1] and Orestis Telelis[2]

[1] Dept. of Informatics, Athens University of Economics and Business, Greece
markakis@gmail.com
[2] Dept. of Computer Science, The University of Liverpool, United Kingdom
telelis@gmail.com

Abstract. We present our results on Uniform Price Auctions, one of the standard sealed-bid multi-unit auction formats, for selling multiple identical units of a single good to multi-demand bidders. Contrary to the truthful and economically efficient multi-unit Vickrey auction, the Uniform Price Auction encourages strategic bidding and is socially inefficient in general, partly due to a "Demand Reduction" effect; bidders tend to bid for fewer (identical) units, so as to receive them at a lower uniform price. Despite its inefficiency, the uniform pricing rule is widely popular by its appeal to the natural anticipation, that identical items should be identically priced. Application domains of its variants include sales of U.S. Treasury bonds to investors, trade exchanges over the internet facilitated by popular online brokers, allocation of radio spectrum licenses etc. In this work we study equilibria of the Uniform Price Auction in undominated strategies. We characterize a class of undominated pure Nash equilibria and quantify the social inefficiency of pure and (mixed) Bayes-Nash equilibria by means of bounds on the Price of Anarchy.

1 Introduction

We study *Uniform Price Auctions*, a standard *Multi-Unit Auction* format, for allocating multiple units of a single good to multi-demand bidders within a single auction process. Multi-unit auctions are deployed in a variety of diverse trade exchanges, including online sales over the internet held by various brokers [20], allocation of radio spectrum licenses [17], sales of U.S. Treasury bonds to investors [22], and allocation of advertisement slots on internet sites [8]. The particular feature of the Uniform Price Auction is a single price for every unit allocated to any bidder; this makes it a proper representative of a wider category of uniform pricing auctions, as opposed to *discriminatory pricing* ones, that sell identical units of a single item at different prices [20, 13]). As observed by Milgrom in [17], resurgence of interest in auction design is owed to a large extent to the success of multi-unit and – particularly – uniform price auction

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formats. Uniform pricing appeals to the intuitive anticipation of identical prices for identical items and eases proxy agents that bid on behalf of their employers; they do not have to explain why they payed more than their competitors.

The design of *mechanisms* for auctioning multiple units of a single good to multi-demand bidders dates back to the seminal work of Vickrey [23]. Since then three *standard* sealed-bid auction formats have been identified in Auction Theory [13]: the Multi-Unit Vickrey Auction, the Uniform Price Auction, and the *Discriminatory Price* Auction. A significant volume of research has been dedicated to identifying the properties of these standard formats [19, 9, 1, 21, 3]. All three auctions have the same bidding format and allocation rule, and have been studied extensively for bidders with *"downward sloping"* (*symmetric submodular* [14]) valuations; these prescribe that the *marginal* value that a bidder has for each additional unit is non-increasing. Each bidder is asked to issue such a non-increasing sequence of *marginal* bids for the k available units. The k highest marginal bids win the auction and each winning bid grants its issuing bidder a distinct unit. The Multi-Unit Vickrey auction charges according to an instance of the Clarke payment rule [6] and generalizes the celebrated single-item Second-Price Auction to the case of multiple units. The Discriminatory Price Auction charges the winning bids as payments thus generalizing the First-Price Auction. The Uniform Price Auction, which was proposed by Friedman [10], charges per allocated unit the highest rejected (losing) marginal bid. The multi-unit Vickrey Auction for submodular bidders optimizes the Social Welfare and is truthful (it is a –weakly – dominant strategy for every bidder to report his marginal values truthfully). Neither the Discriminatory nor the Uniform Price auctions are truthful; they encourage strategic bidding.

In fact, a particular form of strategic bidding in Uniform Price Auctions has been identified as the *Demand Reduction* effect, observed in [19, 9] and formalized in a general model for multi-unit auctions by Ausubel and Cramton [1]. Bidders may choose to shade their marginal bids for some units, only to win fewer ones at a lower uniform price. This leads to diminished revenue and inefficient allocations at equilibrium. In particular it is known that the socially optimal allocation cannot be generally implemented in an equilibrium in (weakly) *undominated strategies*. Despite this effect, variants of Uniform Price Auctions have seen extensive applications, contrary to the Vickrey auction, which has been largely overlooked in practice; implementations of variants of the standard format are offered by several online brokers [3] [20, 12] and are also being used for sales of U.S. Treasury notes to investors since 1992 [22]. We also note that the Uniform Price Auction does retain some interesting characteristics: overbidding any marginal value is a weakly dominated strategy, and so is any misreport of the marginal bid for the *first* unit.

Contribution. We study pure Nash and (mixed) Bayes-Nash equilibria of the Uniform Price Auction in *undominated* strategies. We give a detailed description of (pure) undominated strategies in the standard model of Uniform Price

[3] Among them, `eBay` ceased its own variant in 2009.

Auctions for submodular bidders (Section 4) and demonstrate how their properties follow from a standard assumption, i.e., that bidders issue non-increasing marginal bids for additional units. Although these properties are mentioned or partially derived in previous works, our analysis aims at clarifying some ambiguity between assumptions and implications. Additionally, we give a proposition describing a subset of pure Nash equilibria in undominated strategies.

In Section 5 we study the inefficiency of pure Nash equilibria (PNE) of the Uniform Price Auction in undominated strategies, i.e., the Price of Anarchy (PoA) over the subset of such equilibria. We derive an upper bound of $\frac{e}{e-1}$ for submodular valuation functions. We note here that the auction does have a socially optimal equilibrium (discussed in Section 3, but not in undominated strategies; all undominated PNE are known to be socially inefficient). As noted earlier, this is largely due to the *Demand Reduction* effect [1], whereby a bidder shades his bids for additional units, so as to pay a lower price for the units he wins. Our analysis can be viewed as a quantification of this effect. For any number of units $k \geq 9$, we provide an almost matching lower bound, equal to $\left(1 - e^{-1} + \frac{2}{k}\right)^{-1}$. In Section 6 we consider (mixed) Bayes-Nash equilibria in the *incomplete information* model of Harsanyi. For Bayes-Nash equilibria that emerge from randomized bidding strategy profiles containing only undominated pure strategies in their support, we upper bound the Price of Anarchy by $O(\log k)$.

2 Related Work

Uniform Price Auctions have received extensive study within the economics community. Noussair [19] and Engelbrecht-Wiggans and Kahn [9] gave characterizations of pure Bayes-Nash equilibria under independent private values of bidders, drawn from continuous distributions. They made a first observation of the effect of demand reduction. Ausubel and Cramton formalized demand reduction for a more general model of multi-unit auctions in [1], that allows also interdependent private values. Bresky showed in [3] existence of pure Bayes-Nash equilibria in the independent private values model (with continuous valuation distributions) for several multi-unit auctions, including all three standard formats.

Partly dictated by the practice of auction design and in part because of the computational difficulty of satisfying truthfulness while approximating the social welfare efficiently, there has been a resurgence of interest in the computer science community in studying auction mechanisms that are not necessarily incentive compatible [5, 2, 11, 15]. Our results also follow this line of work of analyzing non-truthful mechanisms. Christodoulou, Kovács and Schapira initialized the study of Combinatorial Auctions, where they proposed that each out of a universe of distinct goods is sold separately and simultaneously to all other goods, in a Second-Price auction. For bidders with fractionally subadditive valuations they proved that this scheme recovers at least $\frac{1}{2}$ of the optimal social welfare in Bayesian (mixed) Nash Equilibrium. Bhawalkar and Roughgarden showed a bound of $O(\log m)$ for the Bayesian Price of Anarchy for subadditive valuations and a bound of 2 for the PoA of pure Nash equilibria [2]. Hassidim *et al.*

proved welfare guarantees for a similar scheme that incorporated simultaneous First-Price auctions instead. Very recently, Syrgkanis and Tardos studied in [15] sequential First- and Second-Price auctions, motivated by the practical issue that supply may not be available at once. Lucier and Borodin [16] analyzed the social inefficiency at (mixed) Bayes-Nash equilibrium of combinatorial auctions for multiple distinct goods, with *greedy* allocation algorithms. They proved Price of Anarchy bounds fairly comparable to the approximation factors of the greedy allocation algorithms, for the underlying welfare optimization problem.

From the mechanism design perspective, Vickrey designed in [23] the first truthful mechanism for auctioning multiple units "in one go", so as to maximize the social welfare. Since then, computationally efficient truthful approximation mechanisms for multi-unit auctions and multi-demand bidders were given by Mu'alem and Nisan in [18] and by Dobzinski and Nisan in [7], even for general valuation functions. Very recently, Vöcking gave a randomized universally truthful polynomial-time approximation scheme for bidders with general valuations [24] (a universally truthful mechanism is a probability distribution over deterministic truthful mechanisms), thus almost closing the problem. In these works, the bids are elicited by the allocation algorithms through polynomially many *value queries* to the bidders, for specific bundles (with the exception of k-minded bidders, whose valuation function has a succinct representation).

3 Model and Definitions

We consider auctioning k units of a single item to a set $\mathcal{N} = [n]$ of n bidders indexed by $i = 1, \ldots, n$. Every bidder $i \in \mathcal{N}$ has a private valuation defined over the quantity of units he receives i.e. $v_i : [k] \mapsto \Re^+$, where $v_i(0) = 0$ and each v_i is non-decreasing. In this work we consider submodular valuation functions:

Definition 1. *A valuation function $f : [k] \mapsto \Re^+$ is called (symmetric)* **submodular** *if for every $x < y$, $f(x) - f(x-1) \geq f(y) - f(y-1)$.*

The following is a well known fact concerning submodular valuations.

Proposition 1. *Given $x, y \in [k]$ with $x \leq y$, a submodular valuation function f satisfies $f(x)/x \geq f(y)/y$.*

A valuation function v_i can be specified by a vector $(m_i(1), ..., m_i(k))$ of the *marginal values* $m_i(j) = v_i(j) - v_i(j-1)$ incurred to bidder i, for each additional unit in his allocation (if v_i is submodular, $m_i(j) \geq m_i(j+1)$).

Uniform Price Auction. In the standard Uniform Price Auction, bidders are asked to submit non-increasing marginal bids. Every bidder i is expected to declare his whole valuation curve as a vector $\boldsymbol{b}_i = (b_i(1), b_i(2), \ldots, b_i(k))$, with $b_i(1) \geq b_i(2) \geq \cdots \geq b_i(k)$, where $b_i(j)$ is the declared marginal value of i for obtaining the j-th unit. A declared bid $b_i(j)$ may differ from the actual marginal value $m_i(j)$. Given a bidding configuration $\boldsymbol{b} = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n)$, the allocation algorithm produces an allocation $\boldsymbol{x}(\boldsymbol{b}) = (x_1(\boldsymbol{b}), x_2(\boldsymbol{b}), \ldots, x_n(\boldsymbol{b}))$. The

Social Welfare under configuration $\boldsymbol{b}$ equals the bidders' total value for $\boldsymbol{x}(\boldsymbol{b})$:

$$SW(\boldsymbol{b}) = \sum_{i=1}^{n} v_i(x_i(\boldsymbol{b}))$$

The allocation algorithm of the Uniform Price Auction is an instantiation of the greedy algorithm described in [14] and is shown in Figure 1. It allocates the next unit to the next highest bid. Every bidder i pays a uniform price $p(\boldsymbol{b})$ per received unit, which equals the highest rejected bid. If under configuration $\boldsymbol{b}$ bidder i is allocated $x_i(\boldsymbol{b})$ units and the uniform price is $p(\boldsymbol{b})$, i pays a total of $x_i(\boldsymbol{b}) \times p(\boldsymbol{b})$ and derives utility $u_i(\boldsymbol{b}) = v_i(x_i(\boldsymbol{b})) - x_i(\boldsymbol{b}) \times p(\boldsymbol{b})$.

This format is a generalization of the single-item Vickrey auction to the case of multiple units, but it does not retain strategyproofness. It always admits an efficient pure Nash equilibrium though: let $\boldsymbol{x}^* = (x_1^*, ..., x_n^*)$ be an optimal allocation[4] of units to the bidders. Consider the profile $\boldsymbol{b}$ with $\boldsymbol{b}_i = (m_i(1), ..., m_i(x_i^*), 0, ..., 0)$ if $x_i^* \geq 1$ and $\boldsymbol{b}_i = \boldsymbol{0}$ otherwise. It can be shown that this is a Nash equilibrium. However, $\boldsymbol{b}_i = \boldsymbol{0}$ is a weakly dominated strategy for bidders i with $x_i^* = 0$ (Nash equilibria in undominated strategies are also known to exist).

A *demand reduction* effect occurs in undominated equilibria of this auction format. Bidders may have an incentive to understate their marginal increase for the j-th unit onwards, for some $j > 1$ [1]. This induces economic inefficiency to equilibria in undominated strategies. Nonetheless, we show that Uniform Price Auctions approximate the optimal Social Welfare within a constant factor.

1. **Set** $x_i = 0$, **for** $i = 1, \ldots, n$.
2. **For** $j = 1, \ldots, k$ **do:**
   (a) $i^* \leftarrow \arg\max_i b_i(x_i + 1)$
   (b) $x_{i^*} \leftarrow x_{i^*} + 1$
3. **return** $\boldsymbol{x}$

Fig. 1: Allocation Algorithm.

Incomplete Information Setting. Every bidder $i \in \mathcal{N}$ obtains his valuation function from a finite set V_i of valuation functions, through a discrete probability distribution $\pi_i : V_i \mapsto [0,1]$ independently of the rest of the biddders; for any particular $v \in V_i$ we write $v \sim \pi_i$ to signify that it is drawn randomly from distribution π_i. The valuation function of every bidder is *private*. A valuation profile $\boldsymbol{v} = (v_1, \ldots, v_n) \in \mathcal{V} = \times_i V_i$ is drawn from a *publicly known distribution* $\pi = \times_i \pi_i$, $\pi : \mathcal{V} \mapsto [0,1]$. We thus write accordingly $\boldsymbol{v} \sim \pi$.

Every bidder i knows his own valuation function v_i – drawn from V_i according to π_i, but does not know the valuation function $v_{i'}$ drawn by any other bidder $i' \neq i$. Bidder i may only use his knowledge of π to estimate $\boldsymbol{v}_{-i}$. Given the publicly known distribution π, the (possibly mixed) strategy of every bidder is a function of his own valuation v_i, denoted by $B_i(v_i)$. B_i maps a valuation function $v_i \in V_i$ to a *distribution* $B_i(v_i) = B_i^{v_i}$, over all possible bid vectors (strategies) for i. In this case we will write $\boldsymbol{b}_i \sim B_i^{v_i}$, for any particular bid vector $\boldsymbol{b}_i$ drawn from this distribution. We also use the notation $\boldsymbol{B}_{-i}^{\boldsymbol{v}_{-i}}$, to refer

[4] For symmetric submodular valuations the allocation algorithm of the Uniform Price Auction outputs an optimal allocation when bidders bid truthfully.

to the vector of randomized strategies of bidders other than i, under valuation profile $\boldsymbol{v}_{-i}$ for these bidders. A *Bayes-Nash equilibrium* (BNE) is a strategy profile $\boldsymbol{B} = (B_1, \ldots, B_n)$ such that for every bidder i and for every valuation v_i, $B_i(v_i)$ maximizes the utility of i in expectation, over the distribution of the other bidders' valuations $\boldsymbol{w}_{-i}$ *given* v_i, and over the distribution induced by the mixed strategies of the bidders. That is, for every pure strategy $\boldsymbol{c}_i$ of i:

$$\mathbb{E}_{\substack{\boldsymbol{w}_{-i}|v_i,\\ \boldsymbol{b}\sim\boldsymbol{B}^{(v_i,\boldsymbol{w}_{-i})}}}\Big[u_i(\boldsymbol{b})\Big] \geq \mathbb{E}_{\substack{\boldsymbol{w}_{-i}|v_i,\\ \boldsymbol{b}_{-i}\sim\boldsymbol{B}^{\boldsymbol{w}_{-i}}}}\Big[u_i(\boldsymbol{c}_i, \boldsymbol{b}_{-i})\Big]$$

where we use notation $\mathbb{E}_{\boldsymbol{v}}$ and $\mathbb{E}_{\boldsymbol{w}_{-i}|v_i}$ to denote expectation over the distributions π and $\pi(\cdot|v_i)$ (*given* v_i) respectively. Fix a valuation profile $\boldsymbol{v} \in \mathcal{V}$ and consider a (mixed) bidding configuration $\boldsymbol{B}^{\boldsymbol{v}}$, under $\boldsymbol{v}$. The Social Welfare $SW(\boldsymbol{B}^{\boldsymbol{v}})$ under $\boldsymbol{B}^{\boldsymbol{v}}$ is defined in expectation over the bidding profiles chosen by the bidders from their randomized strategies. Then, $\mathbb{E}_{\boldsymbol{v}}[SW(\boldsymbol{B}^{\boldsymbol{v}})]$ is the *expected* Social Welfare in *Bayes-Nash Equilibrium*:

$$\mathbb{E}_{\boldsymbol{v}}[SW(\boldsymbol{B}^{\boldsymbol{v}})] = \mathbb{E}_{\substack{\boldsymbol{v}\sim\pi,\\ \boldsymbol{b}\sim\boldsymbol{B}^{\boldsymbol{v}}}}\Big[\sum_i v_i(x_i(\boldsymbol{b}))\Big]$$

We denote by $\boldsymbol{x}^{\boldsymbol{v}}$ the socially optimal assignment under valuation profile $\boldsymbol{v} \in \mathcal{V}$ and, by slight abuse of notation, $\mathbb{E}_{\boldsymbol{v}}[SW(\boldsymbol{x}^{\boldsymbol{v}})]$ is the expected optimal social welfare. We will study the *Bayesian Price of Anarchy*, i.e. the worst case ratio $\mathbb{E}_{\boldsymbol{v}}[SW(\boldsymbol{x}^{\boldsymbol{v}})]/\mathbb{E}_{\boldsymbol{v}}[SW(\boldsymbol{B}^{\boldsymbol{v}})]$ over all distributions π and Bayes-Nash equilibria $\boldsymbol{B}$.

4 Undominated Equilibria

We study bidders with submodular valuation functions. Following Krishna [13] and Milgrom [17], we consider the standard multi-unit auction format, where bidders submit a vector of non-increasing marginal bids, i.e., encode their actual valuation function in a submodular function[5]. A similar situation occurs in combinatorial auctions with item-bidding [5, 2] wherein bidders encode their valuation functions with additive functions.

Assumption 1 *The strategy space of a bidder i consists of all bidding vectors $\boldsymbol{b}_i$ for which $b_i(1) \geq b_i(2) \geq ... \geq b_i(k)$.*

A direct consequence of Assumption 1 is that, under any strategy profile $\boldsymbol{b}$, the price $p(\boldsymbol{b})$ never exceeds any of the winning bids. Lemmas 1 and 2 below state two well known facts about the Uniform Price Auction with submodular bidders (see e.g. [13, 17]). We state them here to signify that Lemma 1 follows from Assumption 1 and Lemma 2 follows from the assumption *and* from Lemma 1.

Lemma 1. *For bidders with submodular valuations, and for any $j \in [k]$, it is a weakly dominated strategy to declare a bid $b_i(j)$ with $b_i(j) > m_i(j)$.*

[5] This requirement is implementable: the auctioneer can exclude non-conforming bidders. Also, simple examples exhibit its necessity for ensuring individual rationality.

Remark 1. Lemma 1 shows that a weakly undominated strategy in our setting captures a stricter notion of conservative behavior than the "no-overbidding" assumption in recent literature (e.g. [2, 4, 5]). In our setting, no-overbidding would mean that for any r units, $\sum_{j=1}^{r} b_i(j) \leq v_i(r)$.

To distinguish from the usual no-overbidding assumption, we call a bidder i who bids at most $m_i(j)$ for any $j \in [k]$ *conservative with respect to marginal bids.*

Lemma 2. *In an undominated strategy, a bidder with a submodular valuation never declares a bid* $b_i(1) \neq v_i(1)$.

We now give a characterization of a subset of undominated equilibria:

Proposition 2. *Let* $\boldsymbol{b}$ *be a pure Nash equilibrium strategy profile of the Uniform Price Auction in undominated strategies for submodular bidders, with uniform price* $p(\boldsymbol{b})$. *There always exists a pure Nash equilibrium* $\boldsymbol{b}'$ *in undominated strategies, satisfying* $\boldsymbol{x}(\boldsymbol{b}') = \boldsymbol{x}(\boldsymbol{b})$ *and:*

1. $b'_i(x) = m_i(x)$, *for every bidder* i *and every* $x \leq x_i(\boldsymbol{b})$.
2. $p(\boldsymbol{b}') \leq p(\boldsymbol{b})$ *and* $p(\boldsymbol{b}')$ *is either* 0 *or equal to* $v_i(1)$ *for some bidder* i.

5 Inefficiency of Pure Nash Equilibria

This section presents welfare guarantees for pure Nash equilibria of the standard form of the Uniform Price Auction, discussed in the previous section. First we are going to show a general result about upper bounding the Price of Anarchy of pure Nash equilibria. Given a configuration $\boldsymbol{b}$, we will be denoting by $\beta_j(\boldsymbol{b})$, $j = 1, \ldots, k$, the j-th lowest winning bid, so that $\beta_1(\boldsymbol{b}) \leq \beta_2(\boldsymbol{b}) \leq \cdots \leq \beta_k(\boldsymbol{b})$. In this section we will omit an explicit reference to $\boldsymbol{b}$ in this notation, as it will be clear from the context. Instead, we use simply β_j, $j = 1, \ldots, k$.

Lemma 3. *Let* $\boldsymbol{b}$ *denote an undominated pure Nash equilibrium of a Uniform Price Auction for* k *units and* $\boldsymbol{x}(\boldsymbol{b})$ *the corresponding allocation. Let* $\boldsymbol{x}^*$ *be an assignment that maximizes the social welfare. The Price of Anarchy is at most:*

$$PoA \leq \sup_{\boldsymbol{b}} \max_{i: x_i^* - x_i(\boldsymbol{b}) > 0} \left[v_i(x_i^*) \cdot \left(v_i\Big(x_i(\boldsymbol{b})\Big) + \sum_{j=1}^{x_i^* - x_i(\boldsymbol{b})} \beta_j \right)^{-1} \right] \tag{1}$$

The following result quantifies the inefficiency of the standard multi-unit Uniform Price auction for multi-demand bidders with symmetric submodular valuation functions and identifies the impact of *demand reduction* [1].

Theorem 1. *The Uniform Price Auction recovers in an undominated pure Nash equilibrium a fraction of at least* $1 - e^{-1}$ *of the optimal Social Welfare, for multi-demand bidders with symmetric submodular valuations.*

Proof. It suffices to upper bound the social inefficiency of undominated equilibria satisfying the properties of Proposition 2. Let $p(\boldsymbol{b})$ be the uniform price paid under equilibrium $\boldsymbol{b}$. To estimate a lower bound on the Social Welfare of $\boldsymbol{b}$, we consider possible deviations of bidders i with $x_i^* > x_i(\boldsymbol{b})$. At least one such bidder exists, otherwise, $x_i(\boldsymbol{b}) \geq x_i^*$ for every i implies that $\boldsymbol{b}$ is socially optimal.

For every bidder i with $x_i^* > x_i(\boldsymbol{b})$ define $r_i(\boldsymbol{b}) = x_i^* - x_i(\boldsymbol{b})$; for every value $j = 1, \ldots, r_i(\boldsymbol{b})$ there exists a deviation that will grant him j additional units to the ones he already holds under $\boldsymbol{b}$; this is due to the fact that all bidders play marginal bids at most equal to their marginal valuations in $\boldsymbol{b}$. Since a sorting of the marginal values determines $\boldsymbol{x}^*$, every "socially optimal winner" i (with $x_i^* \geq 1$) can feasibly deviate under $\boldsymbol{b}$ so as to obtain at least x_i^* units. If $r_i(\boldsymbol{b}) > 0$, a deviation of i for obtaining any $j = 1, \ldots, r_i(\boldsymbol{b})$ *additional* units will raise the uniform price to exactly β_j (using Proposition 2) and cannot be profitable for i:

$$v_i(x_i(\boldsymbol{b}) + j) - (x_i(\boldsymbol{b}) + j) \cdot \beta_j \leq v_i(x_i(\boldsymbol{b})) - x_i(\boldsymbol{b}) \cdot p(\boldsymbol{b})$$

To simplify notation, we use hereafter x_i for $x_i(\boldsymbol{b})$, p for $p(\boldsymbol{b})$ and r_i for $r_i(\boldsymbol{b})$. Then we deduce for every i with $r_i > 0$:

$$\beta_j \geq \frac{1}{j + x_i} \cdot \Big(v_i(x_i + j) - v_i(x_i)\Big), \text{ for } j = 1, \ldots, r_i \tag{2}$$

We can now proceed to upper bound (1) from Lemma 3, using (2) as follows:

$$\begin{aligned}
v_i(x_i) + \sum_{j=1}^{r_i} \beta_j &\geq v_i(x_i) + \sum_{j=1}^{r_i} \frac{1}{j + x_i} \cdot \Big(v_i(x_i + j) - v_i(x_i)\Big) && (3)\\
&= v_i(x_i) + \sum_{j=1}^{r_i} \left(\frac{j}{j + x_i} \cdot \frac{v_i(x_i + j) - v_i(x_i)}{j}\right) \\
&\geq v_i(x_i) + \frac{v_i(x_i^*) - v_i(x_i)}{x_i^* - x_i} \cdot \sum_{j=1}^{r_i} \frac{j}{j + x_i} && (4)\\
&= v_i(x_i) + \frac{v_i(x_i^*) - v_i(x_i)}{x_i^* - x_i} \cdot \left(x_i^* - x_i - x_i \cdot \sum_{j=1}^{r_i} \frac{1}{j + x_i}\right) \\
&= v_i(x_i^*) - \frac{v_i(x_i^*) - v_i(x_i)}{x_i^* - x_i} \cdot x_i \cdot \sum_{j=1}^{r_i} \frac{1}{j + x_i} && (5)\\
&\geq \left(v_i(x_i^*) - \frac{x_i}{x_i^*} \sum_{j=1}^{r_i} \frac{v_i(x_i^*)}{j + x_i}\right) \geq \left(1 - \frac{x_i}{x_i^*} \int_{x_i}^{x_i^*} \frac{1}{y} dy\right) v_i(x_i^*) && (6)\\
&= \left(1 + \frac{x_i}{x_i^*} \cdot \ln \frac{x_i}{x_i^*}\right) \cdot v_i(x_i^*) \geq (1 - e^{-1}) \cdot v_i(x_i^*) && (7)
\end{aligned}$$

Here (3) occurs by substitution of β_j from (2). (4) follows by submodularity of the valuation functions, particularly that $\frac{v_i(x_i+j) - v_i(x_i)}{j} \geq \frac{v_i(x_i^*) - v_i(x_i)}{x_i^* - x_i}$, for any $j = 1, \ldots, r_i$ where $r_i = x_i^* - x_i$. For (6) we used $\frac{v_i(x_i^*) - v_i(x_i)}{x_i^* - x_i} \leq \frac{v_i(x_i^*)}{x_i^*}$, given $v_i(0) = 0$; we bounded the sum of harmonic terms with the integral, using $\sum_{k=m}^{n} f(k) \leq \int_{m-1}^{n} f(x)dx$, for a monotonically decreasing positive function. We obtain the final result by minimizing $f(y) = 1 + y \ln y$ over $(0, 1)$ for $y = e^{-1}$. The claimed bound for the PoA follows by Lemma 3. $\square$

We will produce an almost matching lower bound for the result of theorem 1, which holds for any number of units $k \geq 9$. We note that for $k = 2, 3$ units, tight bounds of $\frac{4}{3}$ and $\frac{18}{13}$ can be derived by direct manipulation of (3).

Theorem 2. *For any $k \geq 9$, there exist instances where the Uniform Price Auction recovers in an undominated pure Nash equilibrium at most a factor of $(1 - e^{-1} + \frac{2}{k})$ of the optimal social welfare, even for 2 submodular bidders.*

Proof. Consider $k \geq 9$ units and 2 bidders. For $q = \lfloor e^{-1} \cdot k - 1 \rfloor$ (notice that $q \geq 1$) define the valuation functions to be:

$$v_1(x) = x \qquad \text{and} \qquad v_2(x) = \begin{cases} x - q \cdot (H_k - H_{k-x}) & x \leq k - q \\ k - q \cdot (1 + H_k - H_q) & x > k - q \end{cases}$$

where H_m is the m-th harmonic number. Notice that $m_2(x) = 0$ for $x > k - q$. It can be verified that v_2 is symmetric submodular in x; for $x \leq k - q$ we have:

$$v_2(x) = x - q \cdot \Big(H_k - H_{k-x}\Big) = \sum_{j=1}^{x} \Big(1 - \frac{q}{k - j + 1}\Big) = \sum_{j=1}^{x} \frac{r - j + 1}{k - j + 1}$$

where $r = k - q$. Then $\frac{r-j+1}{k-j+1} \leq \frac{r-j+2}{k-j+2} = \frac{r-(j-1)+1}{k-(j-1)+1}$, thus $v_2(x) - v_2(x-1) \leq v_2(x-1) - v_2(x-2)$, for $x \leq k - q$; for $x > k - q$, $v_2(x) = v_2(x-1)$, thus v_2 is submodular. For the socially optimal allocation we grant all units to bidder 1, i.e., $\boldsymbol{x}^* = (k, 0, \ldots, 0)$ and $SW(\boldsymbol{x}^*) = k$. Consider next the configuration $\boldsymbol{b}$ where:

$$b_1(j) = \begin{cases} 1, \text{ for } j \leq q \\ 0, \text{ for } j > q \end{cases} \qquad b_2(j) = \begin{cases} \frac{r-j+1}{k-j+1}, \text{ for } j \leq r = k - q \\ 0, \quad \text{ for } j > r \end{cases}$$

Thus, under $\boldsymbol{b}$, q units are obtained by bidder 1 and $k - q$ units by bidder 2. $\boldsymbol{b}$ is a pure Nash equilibrium; indeed, bidder 2 is essentially truthful and, with a uniform price of 0, obtains the maximum of his utility for the won units. Given that he plays undominated strategies, he may not raise any of his bids further. Player 1 also pays the uniform price of 0, so he does not have incentive to drop any of his units. Should player 1 retain any $j \leq r$ of the $r = k - q$ units held by bidder 2, he would hold a total of $k - r + j$ units at a uniform price $\frac{j}{k-r+j}$; the marginal value gain of j to bidder 1 from the extra units is cancelled out by a total payment equal to j. For the social welfare of $\boldsymbol{b}$ we have:

$$SW(\boldsymbol{b}) = v_1(q) + v_2(r) = k \cdot \Big(1 - \frac{q}{k} \cdot (H_k - H_q)\Big)$$

Then, the Price of Anarchy is at least $k/SW(\boldsymbol{b})$, i.e. at least:

$$\begin{aligned} \Big(1 - \frac{q}{k} \cdot \Big(H_k - H_q\Big)\Big)^{-1} &\geq \left(1 - \frac{e^{-1} \cdot k - 2}{k} \cdot \int_{q+1}^{k} \frac{1}{y} dy\right)^{-1} \\ = \left(1 - \frac{e^{-1} \cdot k - 2}{k} \cdot \ln \frac{k}{\lfloor e^{-1}k - 1 \rfloor + 1}\right)^{-1} &\geq \left(1 - e^{-1} + \frac{2}{k}\right)^{-1} \end{aligned}$$

where we used $H_k - H_q = \sum_{r=q+1}^{k} \frac{1}{r} \geq \int_{q+1}^{k+1} \frac{1}{y} dy \geq \int_{q+1}^{k} \frac{1}{y} dy$, for monotonically decreasing positive functions; the final derivation follows by $q + 1 \leq e^{-1} \cdot k$ and $\lfloor e^{-1}k - 1 \rfloor + 1 \geq e^{-1}k$ □

6 Inefficiency of Bayes-Nash Equilibria

In this section we investigate the social inefficiency of (mixed) Bayes-Nash equilibria. Following [5, 2], to ensure the existence of mixed Bayes-Nash equilibria, we make the assumption of a finite bidding space for bidders, using Remark 1 combined with a sufficiently fine discretization. Just like for pure equilibria, we examine Bayes-Nash equilibria with undominated strategies in their support[6].

We introduce auxiliary notation for the analysis that follows. Recall that for any valuation profile $\boldsymbol{v} \in \mathcal{V}$, $\boldsymbol{x}^{\boldsymbol{v}} = (x_1^{\boldsymbol{v}}, \ldots, x_n^{\boldsymbol{v}})$ is the socially optimal assignment. For any bidder $i \in \mathcal{N}$ let $\mathcal{U}^i \subseteq \mathcal{V}$ denote the subset of valuation profiles $\boldsymbol{v} \in \mathcal{V}$ where $x_i^{\boldsymbol{v}} \geq 1$, i.e., $\mathcal{U}^i = \{\boldsymbol{v} \in \mathcal{V} | x_i^{\boldsymbol{v}} \geq 1\}$; these are the profiles under which i is a "socially optimal winner". Accordingly, define $\mathcal{W}^{\boldsymbol{v}} = \{i | x_i^{\boldsymbol{v}} \geq 1\}$. Given any (pure) bidding profile $\boldsymbol{b}$, we use the *"operator"* $\beta_j(\boldsymbol{b})$, to denote the j-th lowest winning bid in $\boldsymbol{b}$, as in section 5. The following Lemma facilitates the expression of BNE conditions regarding unilateral deviations; it has been proved in a different form and under a different context (for simultaneous single-unit auctions with combinatorial bidders) in [5, 2].

Lemma 4. *For each bidder $i \in \mathcal{N}$ with symmetric submodular valuation v_i, define $\boldsymbol{m}_i^{[j]} = (m_i(1), m_i(2), \ldots, m_i(j), 0, 0, \ldots, 0)$. For any conservative bidding profile $\boldsymbol{b}_{-i}$, and for any number of units j: $u_i(\boldsymbol{m}_i^{[j]}, \boldsymbol{b}_{-i}) \geq v_i(j) - j \cdot \beta_j(\boldsymbol{b}_{-i})$.*

Theorem 3. *The Price of Anarchy of Bayes-Nash Equilibria in Uniform Price Auctions with symmemtric submodular bidders is at most $O(\log k)$.*

Proof. (Sketch) For any Bayes-Nash equilibrium $\boldsymbol{B}$, fix any valuation profile $\boldsymbol{v} \in \mathcal{V}$ and a bidder $i \in \mathcal{W}^{\boldsymbol{v}}$. For $j = 1, \ldots, x_i^{\boldsymbol{v}}$, for any valuation profile $\boldsymbol{w}_{-i} \in$

[6] Such Bayes-Nash equilibria can be shown to exist; moreover the strategies in their support can be shown to be conservative with respect to marginal bids.

$\mathcal{V}_{-i}$ and any strategy $\boldsymbol{b} \sim \boldsymbol{B}_{-i}^{\boldsymbol{w}_{-i}}$, apply Lemma 4. Then take expectation over $\boldsymbol{b}_{-i} \sim \boldsymbol{B}_{-i}^{\boldsymbol{w}_{-i}}$ and, subsequently, over all valuation profiles $\boldsymbol{w}_{-i} \in \mathcal{V}_{-i}$, to obtain:

$$\mathbb{E}_{\boldsymbol{w}_{-i}|v_i}\left[\mathbb{E}_{\boldsymbol{b}_{-i}\sim \boldsymbol{B}_{-i}^{\boldsymbol{w}_{-i}}}[u_i(\boldsymbol{m}_i^{[j]}, \boldsymbol{b}_{-i})]\right] \geq v_i(j) - j \cdot \mathbb{E}_{\boldsymbol{w}_{-i}|v_i}\left[\mathbb{E}_{\boldsymbol{b}_{-i}\sim \boldsymbol{B}_{-i}^{\boldsymbol{w}_{-i}}}[\beta_j(\boldsymbol{b}_{-i})]\right]$$

Because under BNE $\boldsymbol{B}$ bidder i does not have incentive to deviate:

$$\mathbb{E}_{\boldsymbol{w}_{-i}|v_i}\left[\mathbb{E}_{\boldsymbol{b}\sim \boldsymbol{B}^{(v_i,\boldsymbol{w}_{-i})}}[u_i(\boldsymbol{b})]\right] \geq \mathbb{E}_{\boldsymbol{w}_{-i}|v_i}\left[\mathbb{E}_{\boldsymbol{b}_{-i}\sim \boldsymbol{B}_{-i}^{\boldsymbol{w}_{-i}}}[u_i(\boldsymbol{m}_i^{[j]}, \boldsymbol{b}_{-i})]\right]$$

Thus $\frac{1}{j}\mathbb{E}_{\boldsymbol{w}_{-i}|v_i}\left[\mathbb{E}_{\boldsymbol{b}\sim \boldsymbol{B}^{(v_i,\boldsymbol{w}_{-i})}}[v_i(x_i(\boldsymbol{b}))]\right] + \mathbb{E}_{\boldsymbol{w}_{-i}|v_i}\left[\mathbb{E}_{\boldsymbol{b}_{-i}\sim \boldsymbol{B}_{-i}^{\boldsymbol{w}_{-i}}}[\beta_j(\boldsymbol{b}_{-i})]\right] \geq \frac{v_i(j)}{j}$. For any pure strategy $\boldsymbol{c}_i$ of bidder i, $\beta_j(\boldsymbol{b}_{-i}) \leq \beta_j(\boldsymbol{c}_i, \boldsymbol{b}_{-i})$ since the presence of $\boldsymbol{c}_i$ means that more bids are competing to win. Also, by independence of π_i, we have that $\sum_{\boldsymbol{w}_{-i}} \pi(\boldsymbol{w}_{-i}|v_i) = 1$. By submodularity, $\frac{v_i(j)}{j} \geq \frac{v_i(x_i^{\boldsymbol{v}})}{x_i^{\boldsymbol{v}}}$. Then:

$$\frac{1}{j} \cdot \mathbb{E}_{\boldsymbol{w}_{-i}|v_i}\left[\mathbb{E}_{\boldsymbol{b}\sim \boldsymbol{B}^{(v_i,\boldsymbol{w}_{-i})}}[v_i(x_i(\boldsymbol{b}))]\right] + \mathbb{E}_{\boldsymbol{w}}\left[\mathbb{E}_{\boldsymbol{b}\sim \boldsymbol{B}^{\boldsymbol{w}}}[\beta_j(\boldsymbol{b})]\right] \geq \frac{v_i(x_i^{\boldsymbol{v}})}{x_i^{\boldsymbol{v}}}$$

Summing both sides over $j = 1, \ldots, x_i^{\boldsymbol{v}}$, then taking the expectation over the distribution of $\boldsymbol{v} \in \mathcal{U}^i$ and summing over $i \in \mathcal{N}$ yields:

$$\begin{aligned} &\sum_i \sum_{\boldsymbol{v}\in\mathcal{U}^i} \pi(\boldsymbol{v}) \sum_{j=1}^{x_i^{\boldsymbol{v}}} \frac{1}{j} \cdot \mathbb{E}_{\substack{\boldsymbol{w}_{-i}|v_i,\\ \boldsymbol{b}\sim \boldsymbol{B}^{(v_i,\boldsymbol{w}_{-i})}}}\Big[v_i(x_i(\boldsymbol{b}))\Big] + \sum_i \sum_{\boldsymbol{v}\in\mathcal{U}^i} \pi(\boldsymbol{v}) \sum_{j=1}^{x_i^{\boldsymbol{v}}} \mathbb{E}_{\substack{\boldsymbol{w},\\ \boldsymbol{b}\sim \boldsymbol{B}^{\boldsymbol{w}}}}\Big[\beta_j(\boldsymbol{b})\Big] \\ &\geq \sum_i \sum_{\boldsymbol{v}\in\mathcal{U}^i} \pi(\boldsymbol{v}) \sum_{j=1}^{x_i^{\boldsymbol{v}}} \frac{v_i(x_i^{\boldsymbol{v}})}{x_i^{\boldsymbol{v}}} = \sum_{\boldsymbol{v}\in\mathcal{V}} \pi(\boldsymbol{v}) \sum_{i\in\mathcal{W}^{\boldsymbol{v}}} v_i(x_i^{\boldsymbol{v}}) = \mathbb{E}_{\boldsymbol{v}}\Big[SW(\boldsymbol{x}^{\boldsymbol{v}})\Big] \end{aligned} \quad (8)$$

The result follows by upper bounding the first and second summands of the first line of (8) by $(1 + \ln k)\mathbb{E}_{\boldsymbol{v}}[SW(\boldsymbol{B}^{\boldsymbol{v}})]$ and $\mathbb{E}_{\boldsymbol{w}}[SW(\boldsymbol{B}^{\boldsymbol{w}})]$ respectively. The bounding of the second summand in particular can be carried out by usage of $\sum_i \sum_{j=1}^{x_i} \beta_j(\boldsymbol{b}) \leq SW(\boldsymbol{b})$, for any bidding configuration $\boldsymbol{b}$ that is conservative w.r.t. marginal bids and for any assignment $\boldsymbol{x}$ of all k units to n bidders. □

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