Nash Equilibria via Polynomial Equations

Richard J. Lipton[1,2] and Evangelos Markakis[1]

[1] Georgia Institute of Technology, Atlanta GA 30332, USA,
{rjl,vangelis}@cc.gatech.edu
[2] Telcordia Research, Morristown NJ 07960, USA

Abstract. We consider the problem of computing a Nash equilibrium in multiple-player games. It is known that there exist games, in which all the equilibria have irrational entries in their probability distributions [19]. This suggests that either we should look for symbolic representations of equilibria or we should focus on computing approximate equilibria. We show that every finite game has an equilibrium such that all the entries in the probability distributions are algebraic numbers and hence can be finitely represented. We also propose an algorithm which computes an approximate equilibrium in the following sense: the strategies output by the algorithm are close with respect to l_∞-norm to those of an exact Nash equilibrium and also the players have only a negligible incentive to deviate to another strategy. The running time of the algorithm is exponential in the number of strategies and polynomial in the digits of accuracy. We obtain similar results for approximating market equilibria in the neoclassical exchange model under certain assumptions.

1 Introduction

Noncooperative game theory has been extensively used for modeling and analyzing situations of strategic interactions. One of the dominant solution concepts in noncooperative games is that of a Nash equilibrium [19]. Briefly, a Nash equilibrium of a game is a situation in which no agent has an incentive to unilaterally deviate from her current strategy. A nice property of this concept is the well known fact that every game has at least one such equilibrium [19].

In this paper we consider the problem of computing a Nash equilibrium in finite games. The proof given by Nash for the existence of equilibria is based on Brouwer's fixed point theorem and is nonconstructive. A natural algorithmic question is whether a Nash equilibrium can be computed efficiently. Even for a 2-player game there is still no polynomial time algorithm. The running time of the known algorithms (see among others [12–16]) is either exponential or has not been determined yet (and is believed to be exponential). For m-person games, $m > 2$, the problem seems to be even more difficult [18]. Recently it has also been shown that finding equilibria with certain natural properties (e.g. maximizing payoff) is **NP**-hard [4, 8]. The complexity of finding a single equilibrium has been addressed as one of the current challenges in computational complexity [20].

An issue related to the complexity of the problem is that even for 3-player games, there exist examples [19] in which the payoff data are rational numbers but all the Nash equilibria have irrational entries. Hence it is still not clear whether an equilibrium can be finitely represented on a Turing machine.

The problems mentioned above suggest two potential directions for research. The first one (perhaps more interesting from a theoretical point of view) is to see whether there exist alternative symbolic representations of Nash equilibria. Symbolic representations of numbers have been used in many areas of mathematics such as algebra or algebraic geometry as well as in algorithmic problems involving symbolic computations. A second, more practical goal, is to look for *approximate* equilibria. An approximate equilibrium is usually defined in the literature as a set of strategies such that, either no player can increase her payoff by a nonnegligible amount if she deviates to another strategy, or the strategies, when seen as probability vectors, are close with respect to some norm to an exact Nash equilibrium.

We will address both objectives by using the observation that Nash equilibria are essentially the roots of a single polynomial equation. In particular we show that every game has at least one Nash equilibrium for which all the entries are algebraic numbers, hence it can be finitely represented. The current bounds for the size of the representation are exponential. We also use results from the existential theory of reals and propose an algorithm for computing an approximate equilibrium in time $poly(\log 1/\epsilon, L, m^n)$. Here m is the number of players, n is the total number of available strategies, ϵ is the degree of approximation and L is the maximum bit size of the payoff data. We show that for the case of two players we can compute an exact Nash equilibrium in time $2^{O(n)}$. This is yet another exponential algorithm for computing an equilibrium in 2-person games. We also note that similar algorithms can be obtained for computing market equilibria under certain assumptions.

1.1 Related work

Recent algorithms for approximate equilibria but only for special classes of games have been obtained in [10, 11]. The fact that Nash equilibria are fixed points of a certain map [19] gives rise to many algorithmic approaches that are based on Scarf's algorithm [22], which is a general algorithm for approximating fixed points. The worst case complexity of this algorithm is exponential in both the total number of strategies and the digits of accuracy [9]. A recent algorithm for approximate equilibria in m-player games with a provable upper bound on the running time is that of [17]. The running time is subexponential in the number of strategies and exponential in the accuracy parameter and the number of players. It is better than ours for games with a small number of players. Our algorithm is better in terms of the dependence on the digits of accuracy and for games with relatively small total number of strategies. Our result is also stronger in the sense that not only players have very small incentive to deviate from the approximate equilibrium, but also the set of strategies which are output are exponentially close to some exact Nash equilibrium. This is not ensured by the algorithms of [22] and [17]. More information on algorithmic approaches can be found in the surveys [18, 25].

The algebraic characterization of Nash equilibria as the set of solutions to a system of polynomial inequalities has been used before. In [24], algebraic techniques are presented for counting the number of completely mixed equilibria. In [5] it is shown that every real algebraic variety is isomorphic to the set of completely mixed Nash equilibria of some three-person game. However representation and complexity issues are not addressed there.

2 Notation and Definitions

2.1 Nash equilibria

Consider a game with m players. Suppose that the number of available (pure) strategies for player i is n_i. Let $n_0 = \max n_i$. An m-dimensional payoff matrix A^i is associated with each player. If players $1, \cdots, m$ play the pure strategies $j_1, \cdots, j_m$ respectively, player i receives a payoff equal to $A^i(j_1, \cdots, j_m)$. For simplicity we assume that the entries of the matrices are integers, at most L bits long and $H = 2^L$ is their maximum absolute value.

A *mixed strategy* for player i is a probability distribution over the set of her pure strategies and will be represented by a vector $x_i = (x_{i1}, x_{i2}, \cdots, x_{i,n_i})$, where $x_{ij} \geq 0$ and $\sum x_{ij} = 1$. Here x_{ij} is the probability that the player will choose her jth pure strategy. The *support* of x_i ($Supp(x_i)$) is the set $\{j : x_{ij} > 0\}$. We will denote by $\mathcal{S}_i$ the strategy space of player i, i.e., the $(n_i - 1)$-dimensional unit simplex. For an m-tuple of mixed strategies $x = (x_1, \cdots, x_m) \in \mathcal{S}_1 \times \cdots \times \mathcal{S}_m$, the expected payoff to the ith player is:

$$P^i(x) = \sum_{j_1=1}^{n_1} \cdots \sum_{j_m=1}^{n_m} A^i(j_1, \cdots, j_m) x_{1,j_1} \cdots x_{m,j_m} \tag{1}$$

Following standard notation, for a tuple of mixed strategies $x = (x_1, \cdots, x_m)$, we will denote by x^{-i} the set of strategies: $\{x_j : j \neq i\}$. We will also denote by (x^{-i}, x'_i) the tuple $(x_1, \cdots, x_{i-1}, x'_i, x_{i+1}, \cdots, x_m)$, i.e., the ith player switches to the strategy x'_i while all other players keep playing the same strategy as in x.

The notion of a Nash equilibrium [19] is formulated as follows:

Definition 1. *A tuple of strategies $x = (x_1, \cdots, x_m) \in \mathcal{S}_1 \times \cdots \times \mathcal{S}_m$ is a Nash equilibrium if for every player i and for every mixed strategy $x'_i \in \mathcal{S}_i$, $P^i(x^{-i}, x'_i) \leq P^i(x)$.*

The definition states that x is a Nash equilibrium if no player has an incentive to unilaterally defect to another strategy. It is easily seen that it is enough to consider only deviations to pure strategies. For a player i, let s_i^j denote her jth pure strategy. Then an equivalent definition is the following: x is a Nash equilibrium if for any player i and any pure strategy of player i, s_i^j: $P^i(x^{-i}, s_i^j) \leq P^i(x)$.

Similarly we can formalize the notion of an ϵ-Nash equilibrium (or simply ϵ-equilibrium), in which players have only a small incentive to deviate:

Definition 2. *For $\epsilon \geq 0$, a tuple of strategies $x = (x_1, \cdots, x_m)$ is an ϵ-Nash equilibrium if for every player i and for every pure strategy s_i^j, $P^i(x^{-i}, s_i^j) \leq P^i(x) + \epsilon$.*

Another notion of approximation is that of ϵ-closeness:

Definition 3. *A point $x = (x_1, \cdots, x_m) \in \mathcal{S}_1 \times \cdots \times \mathcal{S}_m$ is ϵ-close to a point $y \in \mathcal{S}_1 \times \cdots \times \mathcal{S}_m$ if $||x_i - y_i||_\infty \leq \epsilon$ forall $i = 1, ..., m$*

Note that an ϵ-equilibrium is not necessarily close to a real Nash equilibrium.

2.2 Algebra

We give some definitions of basic algebraic concepts that we are going to use in the later sections. For a more detailed exposition we refer the reader to [3].

Definition 4. *A real number α is an algebraic number if there exists a univariate polynomial P with integer coefficients such that $P(\alpha) = 0$.*

Definition 5. *An ordered field R is a real closed field if*

1. *every positive element $x \in R$ is a square (i.e. $x = y^2$ for some $y \in R$).*
2. *every univariate polynomial of odd degree with coefficients in R has a root in R.*

Obviously the real numbers are an example of a real closed field.

3 Nash equilibria are roots of a polynomial

A Nash equilibrium is a solution of the following system of polynomial inequalities and equalities:

$$
\begin{aligned}
& x_{ij} \geq 0 && i = 1, ..., m, j = 1, ..., n_i \\
& \sum_{j=1}^{n_i} x_{ij} = 1 && i = 1, ..., m \\
& P^i(x^{-i}, s_i^j) \leq P^i(x) && i = 1, ..., m, j = 1, ..., n_i
\end{aligned}
\tag{2}
$$

Let $n = \sum n_i$. The system has n variables and $2n + m = O(n)$ multilinear constraints. By adding slack variables we can convert every constraint to an equation, where each polynomial is of degree at most m (the degree of a polynomial is the maximum total degree of its monomials). Note that the slack variables are squared so that we do not have to add any more constraints for their nonnegativity:

$$
\begin{aligned}
B_{ij} &= x_{ij} - \beta_{ij}^2 = 0 && i = 1, ..., m, j = 1, ..., n_i \\
\Gamma_i &= \sum_{j=1}^{n_i} x_{ij} - 1 = 0 && i = 1, ..., m \\
\Delta_{ij} &= P^i(x) - P^i(x^{-i}, s_i^j) - \delta_{ij}^2 = 0 && i = 1, ..., m, j = 1, ..., n_i
\end{aligned}
\tag{3}
$$

We can now combine all the polynomial equations into one by taking the sum of squares ($P_1 = 0$ and $P_2 = 0$ is equivalent to $P_1^2 + P_2^2 = 0$). Therefore we have the following polynomial which we will refer to as the polynomial of the game $(A^1, ..., A^m)$:

$$
\Phi(A^1, ..., A^m) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} B_{ij}^2 + \sum_{i=1}^{m} \Gamma_i^2 + \sum_{i=1}^{m} \sum_{j=1}^{n_i} \Delta_{ij}^2 \tag{4}
$$

Claim. $\Phi(A^1, ..., A^m)$ has degree $2m$, $O(n)$ variables, $n_0^{O(m)}$ monomials and maximum absolute value of its coefficients $O(nH^2)$.

3.1 Finite Representation of Nash Equilibria

Irrationality is not necessarily an obstacle towards obtaining a finite representation for a Nash equilibrium. For example, a real algebraic number α can be uniquely specified by the irreducible polynomial with integer coefficients, P, for which $P(\alpha) = 0$ and an interval which isolates the root α from the other roots of P. In the next Theorem we show that every game has a Nash equilibrium that can be finitely represented. The proof is based on a deep result from the theory of real closed fields known as the transfer principle [3]. We also need to use the fact that equilibria always exist. We are not aware if there is an alternative way of proving Theorem 1. The original topological proof of existence by Nash via Brouwer's fixed point theorem, though powerful enough to guarantee an equilibrium, does not seem to give any further information on the algebraic properties of the equilibria.

Theorem 1. *For every finite game there exists a Nash equilibrium $x = (x_1, ..., x_m)$ such that every entry in the probability distributions $x_1, \cdots, x_m$ is an algebraic number.*

Proof. Given a game $(A^1, ..., A^m)$, the set of its Nash equilibria is the set of roots of the corresponding polynomial Φ (excluding the slack variables). By Nash's proof [19] we know that the equation $\Phi(A^1, ..., A^m) = 0$ has a solution over the reals. Consider the field of the real algebraic numbers R_{alg}. It is known that R_{alg} is a real closed field [3]. The Tarski-Seidenberg theorem, also known as the transfer principle (see [3]), states that for two real closed fields R_1, R_2 such that $R_2 \subseteq R_1$, a polynomial with coefficients in R_2 has a root in R_2 if and only if it has a root in R_1. The real numbers form a real closed field which contains R_{alg}. Since the coefficients of Φ are integers, it follows immediately that there exists a Nash equilibrium in R_{alg}.

A natural question is whether there are reasonable upper bounds for the degree and the coefficient size of the polynomials that represent the entries of an equilibrium. The known upper bounds are exponential. In particular, it follows by [3][Chapter 13] and by the Claim in Section 3 that the degrees of the polynomials will be $m^{O(n)}$ and the coefficient size will be $O(L + \log n)m^{O(n)}$.

3.2 Algorithmic Implications

A more practical goal to pursue is to compute an approximate equilibrium. For this we will use as a subroutine a decision algorithm for the existential theory of reals.

A special case of the decision problem for the existential theory of reals is to decide whether the equation $P(x_1, ..., x_k) = 0$ has a solution over the reals. Here P is a polynomial in k variables of degree d and with integer coefficients. The best upper bound for the complexity of the problem is $d^{O(k)}$, as provided by the algorithms of Basu et al. [2] and Renegar [21].

Theorem 2. *For an m-person game, $m \geq 2$, and for $0 < \epsilon < 1$, there is an algorithm which runs in time poly$(\log 1/\epsilon, L, m^n)$ and computes an m-tuple of strategies $x \in \mathcal{S}_1 \times \cdots \times \mathcal{S}_m$ such that:*

1. *x is ϵ/d-close to some Nash equilibrium y, where $d = 2^{m+1}n_0^m H$.*
2. *$|P^i(x) - P^i(y)| < \epsilon/2$ for all $i = 1, ..., m$.*
3. *x is an ϵ-Nash equilibrium.*

To prove Theorem 2, we need the following Lemma:

Lemma 1. *Let $y = (y_1, ..., y_m)$ be a Nash equilibrium. Let $x = (x_1, ..., x_m)$ be Δ-close to y, where $\Delta < 1$. Then:*

1. *x is an ϵ-Nash equilibrium for $\epsilon = 2^{m+1}n_0^m H\Delta$.*
2. *$|P^i(x) - P^i(y)| < \epsilon/2$ for all $i = 1, ..., m$.*

Proof. We give a sketch of the proof. Since x is Δ-close to y, each x_i can be written in the form $x_i = y_i + e_i$, where $e_i = (e_{i1}, ..., e_{i,n_i})$ and $|e_{ij}| \leq \Delta$. For condition 1, we need to prove that for every player i, $P^i(x) \geq P^i(x^{-i}, s_i^j) - \epsilon$, for every pure strategy s_i^j. Fix a pure strategy s_i^j. Then:

$$\begin{aligned} P^i(x) &= \sum_{j_1} \cdots \sum_{j_m} A^i(j_1, ..., j_m)(y_{1,j_1} + e_{1,j_1}) \cdots (y_{m,j_m} + e_{m,j_m}) \\ &= P^i(y) + E_1 + \cdots + E_{2^m - 1} \end{aligned}$$

where each term E_i is an m-fold sum. Since y is a Nash equilibrium we have:

$$P^i(x) \geq P^i(y^{-i}, s_i^j) + \sum E_i = P^i(x^{-i}, s_i^j) + \sum F_i + \sum E_i$$

where each F_i is an $(m-1)$-fold sum similar to the E_i terms. By performing some simple calculations we can actually show that: $|\sum E_i + \sum F_i| \leq \epsilon$. Hence $\sum E_i + \sum F_i \geq -\epsilon$. Due to lack of space we omit the details for the final version. The second claim can also be verified along the same lines.

From now on, let $\mathcal{A}$ be an algorithm that decides whether $P(x_1, ..., x_k) = 0$ has a solution over the reals in time $d^{O(k)}$, for a degree d polynomial P (either the algorithm of [2] or [21] will do).

Proof of Theorem 2 : By Lemma 1, we only need to find an m-tuple x such that x is ϵ/d-close to some Nash equilibrium y. Let $\Phi(A^1, ..., A^m)$ be the corresponding polynomial of the game. By the Claim in Section 3, the time to compute the coefficients of all the monomials of Φ, given the payoff matrices, is $n_0^{O(m)}$ which is $poly(m^n)$. We can now use $\mathcal{A}$ combined with binary search to compute a rational approximation of some root. Suppose we start with the variable x_{11}. We can add two more constraints to Φ expressing the fact that $x_{11} \in [0, 1/2]$. We then run $\mathcal{A}$ for the new polynomial and if the answer is yes we know that there exists an equilibrium with $x_{11} \in [0, 1/2]$. We can replace the constraints that we added with the ones corresponding to $x_{11} \in [0, 1/4]$. If the answer is no then there exists an equilibrium with $x_{11} \in [1/4, 1/2]$, hence we can continue our binary search in that interval. Proceeding in this manner we will find an interval I_{11} with length at most $\epsilon/(n_1 d)$. For this we need to run $O(\log n_1 d/\epsilon) = O(\log 1/\epsilon + m + m \log n + L) = poly(\log 1/\epsilon, L, m, n)$ times the algorithm $\mathcal{A}$. We will then add to Φ the constraints corresponding to $x_{11} \in I_{11}$ and we will go on to the next variable. When we are done with the variable x_{1,n_i-1}, the interval I_{1,n_i} for x_{1,n_i} is also determined. This is because x_{1,n_i} should be equal to $1 - \sum_{j \neq n_i} x_{1j}$, so that x_1 is a probability distribution. Therefore the length of I_{1,n_i} will be at most ϵ/d. Hence by the end of this step we know that we can select a probability distribution x_1 for the first player such that $|x_1 - y_1|_\infty \leq \epsilon/d$ for some Nash equilibrium y. We continue the procedure to determine an interval for every variable x_{ij}. We can then output a rational number in I_{ij} for each variable so as to ensure that $x_1, ..., x_m$ are probability distributions. Note that by the end we have only added $O(n)$ additional slack variables and constraints. Therefore the total running time will be $poly(\log 1/\epsilon, L, m^n)$.

An exact algorithm for 2-person games We can show that for 2-person games we can compute an exact Nash equilibrium using algorithm $\mathcal{A}$ as a subroutine. The crucial observation is that for 2-person games, if we know the support of the Nash equilibrium strategies, the exact strategies can be computed by solving a linear program. This is true because an equilibrium strategy for player 2 equalizes the payoff that player 1 receives for every pure strategy in her support and vice versa. Hence we can write a linear program and compute the Nash equilibrium with the given support since all the constraints are now linear. By adding constraints of the form $x_{ij} = 0$ and by running $\mathcal{A}$ a linear number of times, we can identify the support of some Nash equilibrium.

Theorem 3. *There exists an algorithm that runs in time $2^{O(n)}$ and computes an exact Nash equilibrium.*

Due to lack of space we omit the proof. This is yet another exponential algorithm for computing an equilibrium in 2-person games. An upper bound on the compexity of the problem can be obtained by the naive algorithm that tries all possible pairs of supports for the two players, which is $O(2^n LP_n^n) = 2^{O(n)}$, where LP_n^n is the time to solve a linear program with $O(n)$ variables and $O(n)$ constraints. Our algorithm achieves the same asymptotic bound but is in fact worse since the constant in the exponent is bigger than two. However we would still like to bring it to the attention of the community firstly because it is a different approach that has not been addressed before to the best of our knowledge and secondly because a future improvement in decision algorithms for low degree polynomial equations would directly imply an improvement in our algorithm too.

4 Approximation of economic equilibria

Similar algorithms can be obtained for computing market equilibria in exchange economies as well as in other economic models. We will briefly mention a special case of the neoclassical exchange model. More information on the general model can be found in [23].

Consider a market of m agents and n commodities (or goods). Each agent has an initial endowment $e_i \in R_+^n$. A continuous, strictly concave and increasing utility function $u_i : R_+^n \to R_+$ is associated with each agent.

Given a price vector $p \in R_+^n$, there exists a unique allocation of goods x to each agent i that maximizes her happiness subject to her spending constraints ($px \leq pe_i$). Given a price vector p and an agent i we denote by $S_i(p)$ the desired allocation:

$$S_i(p) = (S_{i1}(p), ..., S_{in}(p)) = arg \max_{x \in R_+^n} u_i(x) \text{ s. t. } px \leq pe_i \tag{5}$$

For commodity j, let $D_j(p)$ be the total demand for j, i.e., $D_j(p) = \sum_i S_{ij}(p)$. Finally $D(p) = (D_1(p), ..., D_n(p))$ will be called the demand function. We will make the assumption that for each commodity j the demand $D_j(p_1, ..., p_n)$ is a polynomial of degree d.

A price vector at which the market *clears* (goods can be exchanged such that all the agents maximize their total happiness) is called a *market equilibrium*. It is easy to see that such a vector will satisfy the conditions: $p \geq 0$, $D(p) - \sum e_i \leq 0$, $p(D(p) - \sum e_i) = 0$.

Without loss of generality we can assume that the price vector lies on a simplex, i.e., $\sum p_i = 1$. That such an equilibrium always exists follows from the celebrated Arrow-Debreu theorem [1], which in turn is based on Kakutani's fixed point theorem. By using the same argument as in Theorem 1 we can show that there is always an equilibrium in which all the prices are algebraic numbers.

Concerning the complexity of the problem, since all the equations above involve polynomials of degree at most $d+1$ we have that:

Theorem 4. *For any $\epsilon > 0$, there is an algorithm that runs in time poly*$(\log 1/\epsilon$, $d^n)$*, and computes a vector p such that p is ϵ-close to a market equilibrium.*

This improves the bound that can be obtained by Scarf's algorithm, which is exponential in both $\log 1/\epsilon$ and n. More efficient algorithms for market equilibria have been recently obtained (see among others [6, 7]) but only for linear utilities.

5 An application: systems of polynomial inequalities

Much of the research on equilibria in economic models has focused on the algorithmic problem of computing an equilibrium. A common approach has been to reduce the question to an already known and studied problem (e.g. fixed point approximations, linear and nonlinear complementarity problems, systems of polynomial equations and many others). In this section we would like to propose an alternative viewpoint and take advantage of the fact that Nash or market equilibria always exist. In particular, if a problem can be reduced to the existence of an equilibrium in a game or market, then we are guaranteed that a solution exists. As an example, we give the following theorem:

Theorem 5. *Let A be a $n \times n$ matrix and a_i be the i-th row of A. Let $S \subseteq \{1, ..., n\}$. Then the following system of inequalities in n variables $x = (x_1, ..., x_n)$*

$$x^T Ax - a_i x \geq 0, \quad i \in S$$

has a nonzero solution. In fact it has a probability distribution as a solution.

Proof. Consider the *symmetric* game (A, A^T). It is known that every symmetric game has an equilibrium in which both players play the same strategy. The inequalities of the system correspond to the constraints that if both players play strategy x, a deviation to a pure strategy i, for $i \in S$ does not make a player better off.

Deciding whether a set of polynomial equations and inequalities has a solution (or a non-trivial solution) has been an active research topic. Similar theorems can be obtained for any system that corresponds to partial constraints for the existence of Nash equilibria or market equilibria. We do not know if an algebraic proof of Theorem 5 is already known. We believe that the existence of equilibria in games and markets can yield a way of providing simple proofs for the existence of solutions in certain systems of polynomial inequalities.

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References

1. Arrow, K., J., Debreu, G.: Existence of an Equilibrium for a Competitive Economy. Econometrica **22** (1954) 265-290
2. Basu, S., Pollack, R., Roy, M., F.: On the Combinatorial and Algebraic Complexity of Quantifier Elimination. Journal of the ACM **43**(6) (1996) 1002-1045
3. Basu, S., Pollack, R., Roy, M., F.: Algorithms in Real Algebraic Geometry. Springer-Verlag (2003)
4. Conitzer, V., Sandholm, T.: Complexity Results about Nash Equilibria. Technical report CMU-CS-02-135 (2002)
5. Datta, R.: Universality of Nash Equilibria. Math. Op. Res. **28**(3) (2003) 424-432
6. Deng, X., Papadimitriou, C., H., Safra S.: On the Complexity of Equilibria. Annual ACM Symposium on the Theory of Computing (2002) 67-71
7. Devanur, N., Papadimitriou, C., H., Saberi A., Vazirani, V., V.: Market Equilibria via a Primal-Dual-Type Algorithm. Annual IEEE Symposium on Foundations of Computer Science (2002) 389-395
8. Gilboa, I., Zemel, E.: Nash and Correlated Equilibria: Some Complexity Considerations. Games and Economic Behavior (1989)
9. Hirsch, M. D., Papadimitriou, C. H., Vavasis, S.A.: Exponential Lower Bounds for Finding Brouwer Fixed Points. Journal of Complexity **5** (1989) 379-416
10. Kearns, M., J., Littman, M., L., Singh, S., P.: Graphical Models for Game Theory. UAI (2001) 253-260
11. Kearns, M., J., Mansour, Y.: Efficient Nash Computation in Large Population Games with Bounded Influence. UAI (2002) 259-266
12. Koller, D., Megiddo, N.: Finding Mixed Strategies with Small Support in Extensive Form Games. International Journal of Game Theory **25** (1996) 73-92
13. Koller, D., Megiddo, N., von Stengel, B.: Efficient Computation of Equilibria for Extensive Two-person Games. Games and Economic Behavior **14**(2) (1996) 247-259
14. Kuhn, H. W.: An Algorithm for Equilibrium Points in Bimatrix Games. Proceedings of the National Academy of Sciences **47** (1961) 1657-1662
15. Lemke, C. E.: Bimatrix Equilibrium Points and Mathematical Programming. Management Science **11** (1965) 681-689
16. Lemke, C. E., Howson, J. T.: Equilibrium Points of Bimatrix Games. Journal of the Society of Industrial and Applied Mathematics **12** (1964) 413-423
17. Lipton, R., J., Markakis, E., Mehta, A.: Playing Large Games using Simple Strategies. ACM Conference on Electronic Commerce (2003) 36-41
18. McKelvey, R., McLennan, A.: Computation of Equilibria in Finite Games. In Amman, H., Kendrick, D., Rust, J. eds Handbook of Computational Economics **1** (1996)
19. Nash, J. F.: Non-Cooperative games. Annals of Mathematics **54** (1951) 286-295
20. Papadimitriou, C., H.: Algorithms, Games, and the Internet. Annual ACM Symposium on the Theory of Computing (2001) 749-753
21. Renegar, J.: On the Computational Complexity and Geometry of the First Order Theory of Reals. Journal of Symbolic Computation **13**(3) (1992) 255-352
22. Scarf, H.: The Approximation of Fixed Points of a Continuous Mapping. SIAM Journal of Applied Mathematics **15** (1967) 1328-1343
23. Scarf, H.: The Computation of Economic Equilibria (in collaboration with T. Hansen). Cowles Foundation Monograph, Yale University Press (1973)
24. Sturmfels, B.: Solving Systems of Polynomial Equations. Regional Conference Series in Mathematics, **97** (2002)
25. von Stengel, B.: Computing Equilibria for two-person Games. In Aumann, R. and Hart, S. eds, Handbook of Game Theory **3** (2002) 1723-1759