New Algorithms for Approximate Nash Equilibria in Bimatrix Games*

Hartwig Bosse[†] Jaroslaw Byrka[‡] Evangelos Markakis[§]

Abstract

We consider the problem of computing additively approximate Nash equilibria in noncooperative two-player games. We provide a new polynomial time algorithm that achieves an approximation guarantee of 0.36392. We first provide a simpler algorithm, that achieves a 0.38197-approximation, which is exactly the same factor as the algorithm of Daskalakis, Mehta and Papadimitriou.This algorithm is then tuned, improving the approximation error to 0.36392. Our method is relatively fast and simple, as it requires solving only one linear program and it is based on using the solution of an auxiliary zero-sum game as a starting point. Finally we also exhibit a simple reduction that allows us to compute approximate equilibria for multi-player games by using algorithms for two-player games.

1 Introduction

The dominant and most well studied solution concept in noncooperative games has been the concept of Nash equilibrium. A Nash equilibrium is a choice of strategies, one for each player, such that no player has an incentive to deviate (unilaterally). In a series of works [10, 5, 3], it was established that computing a Nash equilibrium is PPAD-complete even for two-player games. The focus has since then been on algorithms for approximate equilibria.

In this work we focus on the notion of *additive approximation* and consider the problem of computing approximate Nash equilibria in bimatrix games. Under the usual assumption that the payoff matrices are normalized to be in $[0,1]^{n\times n}$ (where n is the number of available pure strategies), we say that a pair of mixed strategies is an ϵ-Nash equilibrium if no player can gain more than ϵ by unilaterally deviating to another strategy. In [4] it was proved that it is PPAD-complete to find an ϵ-Nash equilibrium when ϵ is of the order $\frac{1}{poly(n)}$. For constant ϵ however, the problem is still open. In [13], it was shown that for any constant $\epsilon > 0$, an ϵ-Nash equilibrium can be computed in subexponential time ($n^{O(\log n/\epsilon^2)}$). As for polynomial time algorithms, it is fairly simple to obtain a 3/4-approximation (see [11] for a slightly better result) and even better a 1/2-approximation [6]. An improved approximation for

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†University of Frankfurt, Germany. Email: bosse@math.uni-frankfurt.de.

‡Institute of Mathematics, EPFL, Lausanne, Switzerland. Partially supported by the EU Marie Curie Research Training Network, Contract Number MRTN-CT-2003-504438-ADONET. Email: Jaroslaw.Byrka@epfl.ch.

§Center for Math and Computer Science (CWI), Amsterdam, the Netherlands. Partially supported by the EU Marie Curie Research Training Network, Contract Number MRTN-CT-2004-504438-ADONET. Email: vangelis@cwi.nl.

$\epsilon = \frac{3-\sqrt{5}}{2} + \zeta \approx 0.38197 + \zeta$ for any $\zeta > 0$ was obtained by Daskalakis, Mehta and Papadimitriou in [7]. Finally, the currently best known approximation factor of 0.3393 was given by Spirakis and Tsaknakis in [15][1] . Their methodology relies on a gradient-based approach and for more details on this and related results we refer the reader to [16].

Our contribution. We provide two new algorithms for approximate Nash equilibria. The first one achieves exactly the same factor as [7] but with a simpler technique. The second one, which is an extension of the first and has a more involved analysis, achieves an improved approximation of 0.36392. Regarding the running time, both algorithms are quite fast and require the solution of a single linear program.

Our technique is inspired by [12] and the fact that we can compute exact Nash equilibria for zero-sum games in polynomial time via linear programming. In particular, [12] have used the equilibria of zero-sum games of the form $(R + \delta Z, -(R + \delta Z))$, for appropriate values of δ, to derive *well-supported* approximate equilibria, which is a stronger notion of approximation. Here R and C are the payoff matrices of the two players and $Z = -(R + C)$. In both of our algorithms we use a similar starting point as follows: we first find an equilibrium (say x^*, y^*) in the zero-sum game $(R - C, C - R)$. If x^*, y^* is not a good solution for the original game, we then fine-tune the solution and the players take turns and switch to some appropriately chosen strategies. The probabilities of switching are chosen such that the final incentives to deviate become the same for both players. As a result, these probabilities are functions of the parameters of the problem. The final part of the analysis then is to choose these functions so as to minimize the approximation error.

The intuition behind using the auxiliary zero-sum game $(R - C, C - R)$ is that a unilateral switch from x^*, y^* that improves the payoff of one player also improves the payoff of the other player, since x^*, y^* is chosen to be an equilibrium with respect to $R - C$. This allows us to estimate upper bounds on the final incentive of both players to deviate, which we can later optimize. We explain this further in the proof of Theorem 1. At the same time, through our analysis, we discover some of the limitations of using such zero-sum games in deriving approximations for general games by showing that our choice of parameters in Section 4 is optimal for the one-round greedy adjustment framework that we consider.

Finally in Section 6, we show a simple reduction that allows us to compute approximate equilibria for games with more than two players by using algorithms for two-player games. We obtain a 0.60205-approximation for three-player games and 0.71533-approximation for four-player games. To the best of our knowledge these are the first nontrivial polynomial time approximation algorithms for multi-player games.

2 Notation and Definitions

Consider a two person game G, where for simplicity the number of available (pure) strategies for each player is n. Our results still hold when the players do not have the same number of available strategies. We will refer to the two players as the row and the column player and we will denote their $n \times n$ payoff matrices by R, C respectively. Hence, if the row player chooses

[1] Interestingly, as noted in [16], a simpler analysis of the algorithm of [15] with a subset of the inequalities that they use in deriving their upper bound also gives a 0.38197 approximation.

strategy i and the column player chooses strategy j, the payoffs are R_{ij} and C_{ij} respectively.

A *mixed strategy* for a player is a probability distribution over the set of his pure strategies and will be represented by a vector $x = (x_1, x_2, ..., x_n)^T$, where $x_i \geq 0$ and $\sum x_i = 1$. Here x_i is the probability that the player will choose his ith pure strategy. The ith pure strategy will be represented by the unit vector e_i, that has 1 in the ith coordinate and 0 elsewhere. For a mixed strategy pair x, y, the payoff to the row player is the expected value of a random variable which is equal to R_{ij} with probability $x_i y_j$. Therefore the payoff to the row player is $x^T R y$. Similarly the payoff to the column player is $x^T C y$.

A Nash equilibrium [14] is a pair of strategies x^*, y^* such that no player has an incentive to deviate unilaterally. Since mixed strategies are convex combinations of pure strategies, it suffices to consider only deviations to pure strategies:

Definition 1 *A pair of strategies x^*, y^* is a Nash equilibrium if:*

(i) For every pure strategy e_i of the row player, $e_i^T R y^* \leq (x^*)^T R y^*$, and

(ii) For every pure strategy e_i of the column player, $(x^*)^T C e_i \leq (x^*)^T C y^*$.

Assuming that we normalize the entries of the payoff matrices so that they all lie in $[0, 1]$, we can define the notion of an additive ϵ-approximate Nash equilibrium (or simply ϵ-Nash equilibrium) as follows:

Definition 2 *For any $\epsilon > 0$, a pair of strategies x^*, y^* is an ϵ-Nash equilibrium iff:*

(i) For every pure strategy e_i of the row player, $e_i^T R y^* \leq (x^*)^T R y^* + \epsilon$, and

(ii) For every pure strategy e_i of the column player, $(x^*)^T C e_i \leq (x^*)^T C y^* + \epsilon$.

In other words, no player will gain more than ϵ by unilaterally deviating to another strategy. Other approximation concepts have also been studied. In particular, [5] introduced the stronger notion of ϵ-well-supported equilibria, in which every strategy in the support set should be an approximate best response. Another stronger notion of approximation is that of being geometrically close to an exact Nash equilibrium and was studied in [8]. We do not consider these concepts here. For more on these concepts, we refer the reader to [12], [16] and [8].

3 A $(\frac{3-\sqrt{5}}{2})$-approximation

In this section, we provide an algorithm that achieves exactly the same factor as in [7], which is $(3 - \sqrt{5})/2$, but by using a different and simpler method. In the next section we show how to modify our algorithm in order to improve the approximation.

Given a game $G = (R, C)$, where the entries of R and C are in $[0, 1]$, let $A = R - C$. Our algorithm is inspired by [12], as mentioned in the Introduction, and is based on solving the zero-sum game $(A, -A)$ and then modifying appropriately the solution, if it does not provide a good approximation. It is well known that zero-sum games can be solved efficiently using linear programming. The decision on when to modify the zero-sum solution depends on a parameter of the algorithm $\alpha \in [0,1]$. We first describe the algorithm parametrically and then show how to obtain the desired approximation.

Algorithm 1
Let $\alpha \in [0,1]$ be a parameter of the algorithm.

1. Compute an equilibrium (x^*, y^*) for the zero-sum game defined by the matrix $A = R - C$.

2. Let g_1, g_2 be the incentive to deviate for the row and column player respectively if they play (x^*, y^*) in the original game (R, C), i.e., $g_1 = \max_{i=1,\ldots,n} e_i^T R y^* - (x^*)^T R y^*$ and $g_2 = \max_{i=1,\ldots,n} (x^*)^T C e_i - (x^*)^T C y^*$. Without loss of generality, assume, that $g_1 \geq g_2$ (the statement of the algorithm would be completely symmetrical if $g_1 < g_2$).

3. Let $r_1 \in \text{argmax}_{e_i}\ e_i^T R y^*$ be an optimal response of the row player to the strategy y^*. Let $b_2 \in \text{argmax}_{e_i}\ r_1^T C e_i$ be an optimal response of the column player to the strategy r_1.

4. Output the following pair of strategies, $(\hat{x}, \hat{y})$, depending on the value of g_1 with respect to the value of α:

$$(\hat{x}, \hat{y}) = \begin{cases} (x^*, y^*), & \text{if } g_1 \leq \alpha \\ (r_1, (1-\delta_2) \cdot y^* + \delta_2 \cdot b_2), & \text{otherwise} \end{cases}$$

   where $\delta_2 = \frac{1-g_1}{2-g_1}$.

Theorem 1 *Algorithm 1 outputs a* $\max\{\alpha, \frac{1-\alpha}{2-\alpha}\}$*-approximate Nash equilibrium.*

Proof : If $g_1 \leq \alpha$ (recall that we assumed $g_1 \geq g_2$), then clearly (x^*, y^*) is an α-approximate Nash equilibrium.

Suppose $g_1 > \alpha$. We will estimate the satisfaction of each player separately. Suppose b_1 is an optimal response for the row player to $\hat{y}$, i.e., $b_1 \in \text{argmax}_{e_i}\ e_i^T R \hat{y}$. The row player plays r_1, which is a best response to y^*. Hence b_1 can be better than r_1 only when the column player plays b_2, which happens with probability δ_2. Formally, the amount that the row player can earn by switching is at most:

$$\begin{aligned} b_1^T R\hat{y} - r_1^T R\hat{y} &= (1-\delta_2)(b_1^T R y^* - r_1^T R y^*) + \delta_2 (b_1^T R b_2 - r_1^T R b_2) \\ &\leq \delta_2 \cdot b_1^T R b_2 \leq \delta_2 = \tfrac{1-g_1}{2-g_1} \end{aligned}$$

The first inequality above comes from the fact that r_1 is a best response to y^* and the second comes from our assumption that the entries of R and C are in $[0,1]$.

Consider the column player. The critical observation, which is also the reason we started with the zero-sum game $(R-C, C-R)$, is that the column player also benefits (when he plays y^*) from the switch of the row player from x^* to r_1. In particular, since (x^*, y^*) is an equilibrium for the zero-sum game $(R-C, C-R)$, the following inequalities hold:

$$(x^*)^T R e_j - (x^*)^T C e_j \geq (x^*)^T R y^* - (x^*)^T C y^* \geq e_i^T R y^* - e_i^T C y^*,\ \forall\ i, j = 1, ..., n \quad (1)$$

If $e_i = r_1$, we get from (1) that $r_1^T C y^* \geq r_1^T R y^* - (x^*)^T R y^* + (x^*)^T C y^*$. But we know that $r_1^T R y^* - (x^*)^T R y^* = g_1$, which implies:

$$r_1^T C y^* \geq g_1 + (x^*)^T C y^* \geq g_1 \quad (2)$$

Inequality (2) shows that any deviation of the row player from x^*, y^*, that improves his payoff, guarantees at least the same gain to the column player as well. We can now use the lower bound of (2) to estimate the incentive of the column player to change his strategy. He plays $\hat{y}$ while he would prefer to play an optimal response to $\hat{x}$ which is b_2. Since b_2 is played with probability δ_2, by switching he could earn:

$$\begin{aligned}\hat{x}^T C b_2 - \hat{x}^T C \hat{y} &= r_1^T C b_2 - r_1^T C \hat{y} \\ &= r_1^T C b_2 - ((1-\delta_2) r_1^T C y^* - \delta_2 \cdot r_1^T C b_2) \\ &= (1-\delta_2)(r_1^T C b_2 - r_1^T C y^*) \\ &\leq (1-\delta_2)(1-g_1) = \delta_2 = \frac{1-g_1}{2-g_1}\end{aligned}$$

The last inequality above follows from (2). The probability δ_2 was chosen so as to equalize the incentives of the two players to deviate in the case that $g_1 > \alpha$. It is now easy to check that the function $(1-g_1)/(2-g_1)$ is decreasing, hence the incentive for both players to deviate is at most $(1-\alpha)/(2-\alpha)$. Combined with the case when $g_1 \leq \alpha$, we get a $\max\{\alpha, \frac{1-\alpha}{2-\alpha}\}$-approximate equilibrium.

□

In order to optimize the approximation factor of Algorithm 1, we only need to equate the two terms, α and $\frac{1-\alpha}{2-\alpha}$, which then gives:

$$\alpha^2 - 3\alpha + 1 = 0 \tag{3}$$

The solution to (3) in the interval $[0, 1]$ is $\alpha = \frac{3-\sqrt{5}}{2} \approx 0.38197$. Note that $\alpha = 1 - 1/\phi$, where ϕ is the golden ratio. Since α is an irrational number, we need to ensure that we can still do the comparison $g_1 \leq \alpha$ to be able to run Algorithm 1 (note that this is the only point where the algorithm uses the value of α). But to test $g_1 \leq 3 - \sqrt{5}/2$, it suffices to test if $(3 - 2g_1)^2 \geq 5$ and clearly g_1 is a polynomially sized rational number. Concerning complexity, zero-sum games can be solved in polynomial time by linear programming. All the other steps of the algorithm require only polynomial time. Therefore, Theorem 1 implies:

Corollary 2 *We can compute in polynomial time a $\frac{3-\sqrt{5}}{2}$-approximate Nash equilibrium for bimatrix games.*

4 An Improved Approximation

In this section we obtain a better approximation of $1/2 - 1/(3\sqrt{6}) \approx 0.36392$ by essentially proposing a different solution in the cases where Algorithm 1 approaches its worst case guarantee. We first give some motivation for the new algorithm. From the analysis of Algorithm 1, one can easily check that as long as g_1 belongs to $[0, 1/3] \cup [1/2, 1]$, we can have a 1/3-approximation if we run the algorithm with any $\alpha \in [1/3, 1/2)$. Therefore, the bottleneck for getting a better guarantee is when the maximum incentive to deviate is in $[1/3, 1/2]$. In this case, we will change the algorithm so that the row player will play a mix of r_1 and x^*. Note that in Algorithm 1, the probability of playing r_1 is either 0 or 1 depending on the value of g_1. This probability will now be a more complicated function of g_1, derived from a certain optimization problem. As for the column player, we again compute b_2 which is now the best response to the *mixture* of r_1 and x^*- not only to r_1. Then we compute an appropriate mixture of b_2 and y^*. Again, the probability of playing b_2 is chosen so as to equate the incentives of the two players to defect. Finally we should note that our modification will be not on $[1/3, 1/2]$ but instead on a subinterval of the form $[1/3, \beta]$, where β is derived from the optimization that we perform in our analysis.

Algorithm 2

1. Compute an equilibrium (x^*, y^*) for the zero-sum game defined by the matrix $A = R - C$.
2. As in Algorithm 1, let g_1, g_2 be the incentive to deviate for the row and column player respectively if they play (x^*, y^*) in the original game, i.e., $g_1 = \max_{i=1,\ldots,n} e_i^T R y^* - (x^*)^T R y^*$ and $g_2 = \max_{i=1,\ldots,n} (x^*)^T C e_i - (x^*)^T C y^*$. Without loss of generality, assume, that $g_1 \geq g_2$.
3. Let $r_1 \in \operatorname{argmax}_{e_i} e_i^T R y^*$ be an optimal response of the row player to the strategy y^*.
4. The row player will play a mixture of r_1 and x^*, where the probability of playing r_1 is given by:
$$\delta_1 = \delta_1(g_1) = \begin{cases} 0, & \text{if } g_1 \in [0, 1/3] \\ \Delta_1(g_1), & \text{if } g_1 \in (1/3, \beta] \\ 1, & \text{otherwise} \end{cases}$$
where $\Delta_1(g_1) = (1 - g_1)\left(-1 + \sqrt{1 + \frac{1}{1-2g_1} - \frac{1}{g_1}}\right)$.
5. Let b_2 be an optimal response of the column player to $((1 - \delta_1)x^* + \delta_1 r_1)$, i.e., $b_2 \in \operatorname{argmax}_{e_i} ((1 - \delta_1)x^* + \delta_1 r_1)^T C e_i$. Let also $h_2 = (x^*)^T C b_2 - (x^*)^T C y^*$, i.e., the gain from switching to b_2 if the row player plays x^*.
6. The column player will play a mixture of b_2 and y^*, where the probability of playing b_2 is given by:
$$\delta_2 = \delta_2(\delta_1, g_1, h_2) = \begin{cases} 0, & \text{if } g_1 \in [0, 1/3] \\ \max\{0, \Delta_2(\delta_1, g_1, h_2)\}, & \text{if } g_1 \in (1/3, \beta] \\ \frac{1-g_1}{2-g_1}, & \text{otherwise} \end{cases}$$
where $\Delta_2(\delta_1, g_1, h_2) = \frac{\delta_1 - g_1 + (1-\delta_1)h_2}{1 + \delta_1 - g_1}$.
7. Output $(\hat{x}, \hat{y}) = ((1 - \delta_1)x^* + \delta_1 r_1, (1 - \delta_2)y^* + \delta_2 b_2)$.

In our analysis, we will take β to be the solution to $\Delta_1(g_1) = 1$ in $[1/3, 1/2]$, which coincides with the root of the polynomial $x^3 - x^2 - 2x + 1$ in that interval and it is:

$$\beta = \frac{1}{3} + \frac{\sqrt{7}}{3}\cos\left(\frac{1}{3}\tan^{-1}\left(3\sqrt{3}\right)\right) - \frac{\sqrt{21}}{3}\sin\left(\frac{1}{3}\tan^{-1}\left(3\sqrt{3}\right)\right) \tag{4}$$

Calculations show $0.445041 \leq \beta \leq 0.445042$. The emergence of β in our analysis is explained in Lemma 3.

Remark 1 *The actual probabilities δ_1 and δ_2 can be irrational numbers (and so is β). However, for any constant $\epsilon > 0$, we can take approximations of high enough accuracy of all the square roots that are involved in the calculations so that the final loss in the approximation ratio will be at most ϵ. From now on, for ease of exposition, we will carry out the analysis of Algorithm 2, as if we can compute exactly all the expressions involved.*

Note that for $g_1 \in [\frac{1}{3}, \frac{1}{2}]$ and $\delta_1 \in [0,1]$ the denominators that appear in the functions Δ_1, Δ_2 do not vanish. The following lemma ensures that $\hat{x}$ is a valid strategy. It will be proved in Section 5. That $\hat{y}$ is also a valid strategy is proved within Lemma 4.

Lemma 3 *For $g_1 \in (1/3, \beta]$ we have $\Delta_1(g_1) \in [0,1]$.*

Now we bound the incentives of players to deviate. Let F be the following function:

$$F(\delta_1, g_1, h_2) := \frac{(\delta_1 (1 - g_1 - h_2) + h_2)(1 - (1 - \delta_1)h_2)}{1 + \delta_1 - g_1} \tag{5}$$

Lemma 4 *The pair of strategies $(\hat{x}, \hat{y})$ is a λ-Nash equilibrium for game (R, C) with*

$$\lambda \leq \begin{cases} g_1 & \text{if } g_1 \leq 1/3 \\ \max_{h_2 \in [0, g1]} \begin{cases} F(\delta_1, g_1, h_2) & \text{if } \Delta_2(\delta_1, g_1, h_2) \geq 0 \\ (1 - \delta_1) g_1 & \text{if } \Delta_2(\delta_1, g_1, h_2) < 0 \end{cases} & \text{if } g_1 \in (1/3, \beta] \\ \frac{1 - g_1}{2 - g_1} & \text{if } g_1 > \beta \end{cases} \tag{6}$$

Proof : In the case that $g_1 \in [0, 1/3] \cup [\beta, 1]$, the answer essentially follows from the proof of Theorem 1. The interesting case is when $g_1 \in [1/3, \beta]$.

Case 1: $g_1 \leq 1/3$
$(\hat{x}, \hat{y}) = (x^*, y^*)$ which is by definition a g_1-approximate Nash equilibrium.

Case 2a: $g_1 \in (1/3, \beta]$ and $\Delta_2(\delta_1, g_1, h_2) \geq 0$
Recall that Lemma 3 implies $\hat{x}$ is a valid strategy in Case 2. Observe, that $\delta_2(g_1, \delta_1, h_2) = \Delta_2(g_1, \delta_1, h_2) = \frac{\delta_1 - g_1 + (1 - \delta_1) h_2}{1 + \delta_1 - g_1} \leq 1$ is a valid probability, and therefore $\hat{y}$ is a valid mixed strategy too.

We estimate the incentive for the row player to deviate from $\hat{x}$. If b_1 is an optimal response to $\hat{y}$, then the gain from switching is at most:

$$\begin{aligned} b_1^T R\hat{y} - \hat{x}^T R\hat{y} = \quad & (b_1 - \hat{x})^T R\hat{y} = \\ = \quad & \delta_2 (b_1 - \hat{x})^T R b_2 & +(1 - \delta_2)(b_1 - \hat{x})^T R y^* \\ \leq \quad & \delta_2 (1 - \hat{x}^T R b_2) & +(1 - \delta_2)(b_1 - \hat{x})^T R y^* \\ = \quad & \delta_2 (1 - \delta_1 r_1^T R b_2 - (1 - \delta_1)(x^*)^T R b_2) & +(1 - \delta_2)(\delta_1 (b_1 - r_1)^T R y^* + (1 - \delta_1)(b_1 - x^*)^T R y^*) \end{aligned}$$

By (1) we have $(x^*)^T R b_2 \geq (x^*)^T C b_2 - (x^*)^T C y^* + (x^*)^T R y^* \geq h_2$. Also r_1 is a best response to y^*, hence $(b_1 - r_1)^T R y^* \leq 0$ and $(b_1 - x^*)^T R y^* \leq g_1$. Therefore, the gain from deviating is at most:

$$b_1^T R\hat{y} - \hat{x}^T R\hat{y} \leq \delta_2 (1 - (1 - \delta_1) h_2) + (1 - \delta_2)(1 - \delta_1) g_1 = \text{EST}_1.$$

We now estimate the incentive of the column player to switch. The best response to $\hat{x}$ for the column player is b_2, which is played with probability δ_2. Thus the incentive to deviate from $\hat{y}$ is:

$$\begin{aligned} \hat{x}^T C b_2 - \hat{x}^T C\hat{y} \quad & = (1 - \delta_2)(\hat{x}^T C b_2 - \hat{x}^T C y^*) \\ & = (1 - \delta_2)((1 - \delta_1)((x^*)^T C b_2 - (x^*)^T C y*) + \delta_1 (r_1^T C b_2 - r_1^T C y^*)) \\ & \leq (1 - \delta_2)((1 - \delta_1) h_2 + \delta_1 (1 - g_1)) = \text{EST}_2 \end{aligned}$$

The last inequality follows from the definitions of g_1 and h_2. It remains to observe that our choice of $\delta_2(\delta_1, g_1, h_2) = \frac{\delta_1 - g_1 + (1-\delta_1)h_2}{1+\delta_1 - g_1}$ makes these estimates both equal to $F(\delta_1, g_1, h_2)$:

$$\mathrm{EST}_1 = \mathrm{EST}_2 = \frac{(\delta_1 (1 - g_1 - h_2) + h_2)(1 - (1 - \delta_1)h_2)}{\delta_1 + 1 - g_1} = F(\delta_1, g_1, h_2).$$

Case 2b: $g_1 \in (1/3, \beta]$ and $\Delta_2(\delta_1, g_1, h_2) < 0$
Then $\hat{y} = y^*$ and the best response of the row player is r_1. Hence he can improve his payoff by at most

$$r_1^T R y^* - \hat{x}^T R y^* = r_1^T R y^* - (\delta_1 \cdot r_1^T R y^* + (1 - \delta_1)((x^*)^T R y^*)) = (1 - \delta_1) g_1$$

while the column player can improve by at most

$$\hat{x}^T C b_2 - \hat{x}^T C y^* = \delta_1 (r_1^T C b_2 - r_1^T C y^*) + (1 - \delta_1)((x^*)^T C b_2 - (x^*)^T C y^*)$$

By (1) we can see that $r_1^T C y^* \geq g_1$. Hence

$$\hat{x}^T C b_2 - \hat{x}^T C y^* \leq \delta_1 (1 - g1) + (1 - \delta_1) h_2$$

It is easy to check that $\Delta_2(g_1, \delta_1, h_2) < 0$ implies $\delta_1(1 - g1) + (1 - \delta_1)h_2 < (1 - \delta_1)g_1$. Therefore the maximum incentive to deviate in this case is at most $(1 - \delta_1)g_1$. Combining Case 2a and Case 2b, and taking the worst possible case over the range of h_2 (recall that $h_2 \leq g_2 \leq g_1$), we get precisely the expression in the statement of Lemma 4.

Case 3: $g_1 > \beta$
Notice that in this case, the players are playing the same strategies as in Algorithm 1, when $g_1 \geq \alpha$. By the analysis in the proof of Theorem 1, we see that the maximum incentive is $(1 - g_1)/(2 - g_1)$. □

We will now argue that our choice of $\Delta_1(g_1)$ is optimal for any $g_1 \in (\frac{1}{3}, \beta]$ and that the expression (6) from Lemma 4 achieves an improvement over Algorithm 1. For this, we need to find the worst possible approximation in Case 2 of Lemma 4. In particular, we need to look at the maxima of the following function:

$$P(g_1) := \min_{\delta_1 \in [0,1]} \max_{h_2 \in [0, g_1]} \begin{cases} F(\delta_1, g_1, h_2) & \text{if } \Delta_2(\delta_1, g_1, h_2) \geq 0 \\ (1 - \delta_1) g_1 & \text{if } \Delta_2(\delta_1, g_1, h_2) < 0 \end{cases} \tag{7}$$

Lemma 5 *The tuple $(\delta_1, h_2) = (\Delta_1(g_1), g_1)$ is an optimal solution for the expression $P(g_1)$. Furthermore, the maximum of $P(g_1)$ over g_1 is $\frac{1}{2} - \frac{1}{3\sqrt{6}}$, i.e., the following holds*

$$P(g_1) = F(\Delta_1(g_1), g_1, g_1) \quad \forall g_1 \in [\frac{1}{3}, \frac{1}{2}] \tag{8}$$

$$\max_{g_1 \in [\frac{1}{3}, \beta]} P(g_1) = \frac{1}{2} - \frac{1}{3\sqrt{6}} \leq 0.36392. \tag{9}$$

The lemma will be proved in Section 5. Given Remark 1, we are now ready to conclude with the following:

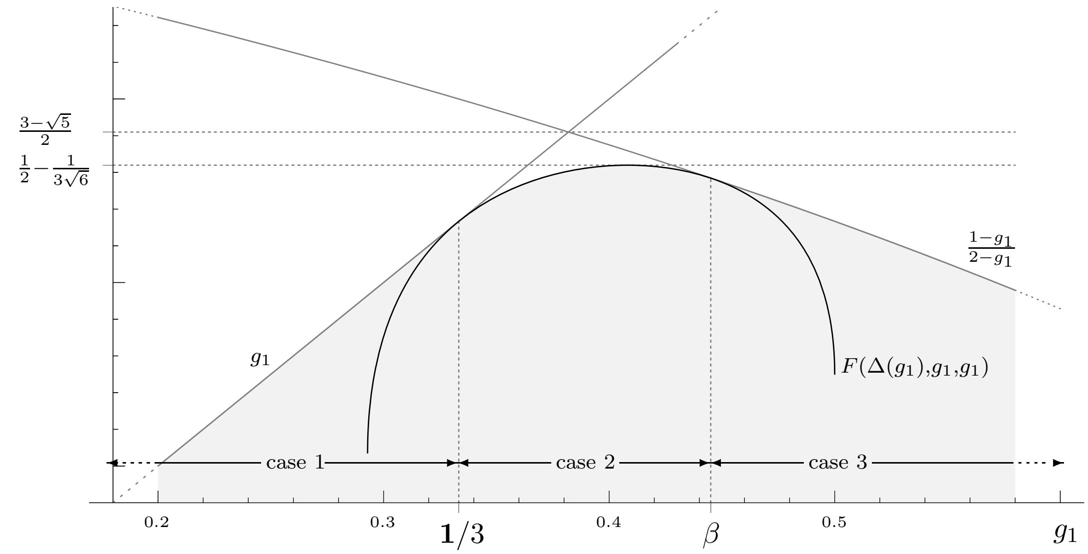

Figure 1: How the approximation factor depends on g_1.

Theorem 6 *For any $\epsilon > 0$, Algorithm 2 computes a $(0.36392 + \epsilon)$-approximate Nash equilibrium.*

Proof : By Lemma 4 the output of Algorithm 2, $(\hat{x}, \hat{y})$ is a pair of mixed strategies for players, such that the incentive of players to deviate is bounded by (6). By Lemma 5 we have that for $g_1 \in (1/3, \beta]$ the expression (6) is bounded by $\frac{1}{2} - \frac{1}{3\sqrt{6}} \leq 0.36392$. It is easy to observe, that for other values of g_1 the expression (6) takes only smaller values. In particular, it is at most 1/3 when $g_1 \in [0, 1/3]$ and at most $\frac{1-\beta}{2-\beta} \approx 0.3569$ when $g_1 > \beta$. The dependence of the approximation on the variable g_1 is presented in Figure 1. □

A Tight Example: The analysis that we have presented is tight. Tracing all inequalities used, we constructed the following worst-case example, on which Algorithm 2 yields a 0.36392-approximate equilibrium:

$$R = \begin{pmatrix} 0 & \alpha & \alpha \\ \alpha & 0 & 1 \\ \alpha & 1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & \alpha & \alpha \\ \alpha & 1 & 1/2 \\ \alpha & 1/2 & 1 \end{pmatrix} \quad \text{where } \alpha = 1/\sqrt{6}.$$

5 Proof of Lemma 3 and Lemma 5

Proof of Lemma 3 :

We show that Δ_1 maps $[1/3, \beta]$ into $[0, 1]$, where Δ_1 (see Algorithm 2) is defined as

$$\Delta_1(g_1) := (1 - g_1)\left(-1 + \sqrt{1 + \frac{1}{1 - 2g_1} - \frac{1}{g_1}}\right).$$

It is easy to check that $\Delta_1(1/3) = 0$. We will show that Δ_1 is real-valued and monotone increasing on the interval $[1/3, 1/2)$. Then we show that $1/3 < \beta < 1/2$, and $\Delta_1(\beta) = 1$.

To check that $\Delta_1(g_1)$ takes real values on $[1/3, 1/2)$, it is easy to verify that the radicand, i.e., the expression under the square root, is nonnegative in this domain.

$$\left(1 + \frac{1}{1-2g_1} - \frac{1}{g_1}\right) \quad \geq \quad 1 \quad \text{for all} \;\; g_1 \in [1/3, 1/2). \tag{10}$$

To check the monotonicity of $\Delta_1(g_1)$, we calculate $\Delta_1'(g_1)$ and find

$$\Delta_1'(g_1) = 1 + \frac{1 - 3g_1 - 2g_1^2 + 14g_1^3 - 8g_1^4}{2(1-2g_1)^2 g_1^2 \sqrt{1 + \frac{1}{1-2g_1} - \frac{1}{g_1}}} \quad > 0 \quad \text{for all} \quad g_1 \in [1/3, 1/2). \tag{11}$$

The inequality in (11) is obtained as follows: Inequality (10) shows that the radicand in (11) is strictly positive on $[1/3, 1/2)$. So the denominator appearing in $\Delta_1'(g_1)$ is real and positive. For the numerator appearing in $\Delta_1'(g_1)$ the following estimation holds for all $g_1 \in [1/3, 1/2)$:

$$\begin{aligned}
1 - 3g_1 - 2g_1^2 + 14g_1^3 - 8g_1^4 \;&=\; \tfrac{1}{2}(3 + g_1 \quad + (1-g_1)(4g_1+1)(\,-2 + (1-2g_1)^2)\;) \\
&\geq\; \tfrac{1}{2}(3 + g_1 \quad + (1-g_1)(4g_1+1)(\,-2\,)\;) \\
&=\; \tfrac{1}{2}\left((1 - \tfrac{5}{2}g_1)^2 + \tfrac{7}{4}g_1^2\right) \quad > 0.
\end{aligned}$$

Here the first inequality holds since $g_1 \in [1/3, 1/2)$ implies $(1-g_1)(4g_1+1) > 0$. This proves (11) showing that Δ_1 is strictly increasing on the interval $[1/3, 1/2)$.

Now we calculate $g \in [1/3, 1/2)$ for which $\Delta_1(g) = 1$ holds. In the following let $x \in [1/3, 1/2)$. This implies $0 < 2 - x$ and $0 < 1 - x$, which together with (10) gives rise to the second equivalence in the following:

$$\begin{aligned}
\Delta_1(x) = 1 \;\;\Leftrightarrow\;\; (2-x) \;&=\; (1-x)\sqrt{1 + \tfrac{1}{1-2x} - \tfrac{1}{x}} \\
\Leftrightarrow\;\; (2-x)^2 \;&=\; (1-x)^2\left(1 + \tfrac{1}{1-2x} - \tfrac{1}{x}\right) \;\;\Leftrightarrow\;\; 1 - 2x - x^2 + x^3 = 0.
\end{aligned}$$

The polynomial $p(x) := 1 - 2x - x^2 + x^3$ has *exactly one* zero in $[1/3, 1/2]$, since p is monotone decreasing on this domain: One calculates $p'(x) = -2x - 2(1 - 3x^2) \leq -2x < 0$ for all $x \in [1/3, 1/2]$. Moreover one has $p(1/3) = 7/27$ and $p(1/2) = -1/8$, showing that p has a root within the interval.

Substituting $x = \frac{1}{3}\left(1 + \sqrt{7}\cos(\alpha) - \sqrt{21}\sin(\alpha)\right)$ and $\alpha = \arctan(t)/3$ leads to

$$1 - 2x - x^2 + x^3 = \frac{7}{27}\left(1 - \sqrt{28}\cos(3\alpha)\right) = \frac{7}{27}\left(1 - \frac{\sqrt{1+27}}{\sqrt{1+t^2}}\right)$$

where the last term is zero for $t = 3\sqrt{3}$. Resubstitution shows that $p(\beta) = 0$ holds for

$$\beta = \frac{1}{3}\left(1 + \sqrt{7}\cos(\alpha) - \sqrt{21}\sin(\alpha)\right)$$

where $\alpha = \frac{1}{3}\arctan\left(3\sqrt{3}\right)$. Taylor expansion of the corresponding terms leads to $0.445041 < \beta < 0.445042$, proving $\beta \in [1/3, 1/2)$. This shows $\Delta_1(\beta) = 1$, which proves the lemma. □

In the proof of Lemma 5 we will make repeated use of the following simple observation:

Fact 7 *The square function is monotone increasing on the positive domain, i.e.,*

$$a - b \geq 0 \Leftrightarrow a^2 - b^2 \geq 0 \quad \textit{holds for all } a, b \in \mathbb{R}, \ \ a, b \geq 0. \tag{12}$$

We solved the univariate minimization problems that arise in Lemma 5 in the classic manner, eventually leading to the minimizer $\Delta_1(g)$. This procedure is lengthy, so here we give an uninspiring but shorter proof. The proof is based on the following Lemma:

Lemma 8 *For every pair* $(g, \delta) \in [1/3, \beta] \times [0, 1]$ *we find*

$$F(\delta, g, g) = \max_{h \in [0,g]} \begin{cases} F(\delta, g, h) & \textit{if } \Delta_2(\delta, g, h) \geq 0 \\ (1-\delta)g & \textit{if } \Delta_2(\delta, g, h) < 0 \end{cases} \tag{13}$$

$$F(\Delta_1(g), g, g) = \min_{d \in [0,1]} F(d, g, g) \tag{14}$$

We postpone the proof of Lemma 8 to the end of this Section.

Proof of Lemma 5 : Combining (13) and (14) from Lemma 8 we obtain

$$F(\Delta_1(g_1), g_1, g_1) = \min_{\delta_1 \in [0,1]} \max_{h_2 \in [0, g_1]} \begin{cases} F(\delta_1, g_1, h_2) & \text{if } \Delta_2(\delta_1, g_1, h_2) \geq 0 \\ (1-\delta_1) g_1 & \text{if } \Delta_2(\delta_1, g_1, h_2) < 0. \end{cases}$$

For ease of exposition, we drop the subscripts of the variables from now on. Hence we are left to prove $\max_{g \in [\frac{1}{3}, \beta]} F(\Delta_1(g), g, g) = \frac{1}{2} - \frac{1}{3\sqrt{6}} \leq 0.36392$ where

$$F(\Delta_1(g), g, g) = \frac{1}{4} - \frac{1}{4}(1-2g)(3-2g)(4g-1) + 2(1-g)\sqrt{g(1-2g)(-1+4g-2g^2)}$$

It is easy to check that (10) implies that the radicand $g(1-2g)(-1+4g-2g^2)$ is nonnegative for all $g \in [1/3, \beta]$. We now prove that the maximum of $F(\Delta(g), g, g)$ on $[\frac{1}{3}, \beta]$ is assumed in $1/\sqrt{6}$: Straightforward calculation leads to

$$\mathcal{F}^* := F\left(\Delta(1/\sqrt{6}) \ , \ 1/\sqrt{6} \ , \ 1/\sqrt{6}\right) = \frac{1}{2} - \frac{1}{3\sqrt{6}} \ .$$

Fixing $g \in [1/3, \beta]$ (arbitrarily), one finds:

$$\mathcal{F}^* - F\left(\Delta_1(g), \ g, \ g\right) = \\ = \underbrace{\frac{1}{4} - \frac{1}{3\sqrt{6}} + \frac{1}{4}(1-2g)(3-2g)(4g-1)}_{\geq 0 \ (*)} - \underbrace{2\,(1-g)\,\sqrt{g(1-2g)(-1+4g-2g^2)}}_{\geq 0 \ (**)} \ .$$

Here $(*)$ and $(**)$ are implied by the choice of g, i.e., $(3-2g) \geq 2(1-g) \geq (1-2g) \geq 0$, and $4g - 1 \geq 1/3 > 0$ hold. Finally since $\sqrt{6} > 2$ we have $\frac{1}{4} - \frac{1}{3\sqrt{6}} > \frac{1}{12} > 0$.

The inequalities in $(*)$ and $(**)$ together with (12) lead to the equivalence

$$\mathcal{F}^* - F\left(\Delta_1(g), \ g, \ g\right) \geq 0 \quad \Leftrightarrow$$

$$\underbrace{\left(\frac{1}{4} - \frac{1}{3\sqrt{6}} + \frac{1}{4}(1-2g)(3-2g)(4g-1)\right)^2 - 4(1-g)^2\left(g(1-2g)(-1+4g-2g^2)\right)}_{= \left(\frac{11}{18} + \frac{2}{3\sqrt{6}}(3-g) + (1-g)^2\right)\left(g - \frac{1}{\sqrt{6}}\right)^2} \geq 0.$$

Here the second inequality holds for the chosen g, since the term can be reformulated as shown under the brace, where $(3-g)>0$ holds by the restriction $g \in [1/3, \beta]$.

Thus we showed $\mathcal{F}^* = F(\Delta_1(1/\sqrt{6}), 1/\sqrt{6}, 1/\sqrt{6}) \geq F(\Delta_1(g), g, g)$, proving the lemma, since $g \in [1/3, \beta]$ was chosen arbitrarily and $1/\sqrt{6} \in [1/3, \beta]$ is implied by $0.40 \leq 1/\sqrt{6} \leq 0.41 < \beta$.

□

It now remains to prove Lemma 8.

Proof of Lemma 8 :

Fix some pair $(g, \delta) \in [1/3, \beta] \times [0, 1]$. We rewrite (13) as

$$F(\delta, g, g) \leq \left(\max_{h \in [0,g]} \begin{cases} F(\delta, g, h) & \text{if } \Delta_2(\delta, g, h) \geq 0 \\ (1-\delta)g & \text{if } \Delta_2(\delta, g, h) < 0 \end{cases} \right) \leq \max_{h \in [0,g]} F(\delta, g, g) \qquad (15)$$

and prove it as follows: Brief calculation together with $(1-g) > 0$ lead to $\Delta_2(\delta, g, g) = (1-g)\delta/(1-g+\delta) \geq 0$. So there is a $h^* \in [0, g]$, namely $h^* := g$, such that $\Delta_2(\delta, g, h^*) \geq 0$. This implies the first inequality in (15).

Observe that to prove the second inequality in (15), it suffices to show that

$$F(\delta, g, g) \geq (1-\delta)g \quad \text{and} \quad F(\delta, g, g) \geq F(\delta, g, h) \quad \text{for all } h \in [0, g] \qquad (16)$$

both hold – independently of the value of Δ_2. Quick calculation proves the first inequality of (16): Recall that the choice on (g, δ) implies $(1-g) \geq 0$, $2\delta g \geq 0$, and $(1-2g) \geq 0$, yielding

$$F(\delta, g, g) - (1-\delta)g = \frac{(1-g)\,\delta}{(1-g)+\delta}\ (2\delta g + (1-2g)\ ) \geq 0.$$

To obtain the second inequality of (16), we show that for the chosen δ, g, the function $F(\delta, g, h)$ is monotone non-decreasing on $h \in [0, g]$: Recalling $h \leq g \leq 1/2$ we find $(1-2h) \geq 0$, implying

$$\frac{\mathrm{d}F(\delta, g, h)}{\mathrm{d}h} = \frac{(1-2h)(1-\delta)^2 + g\delta(1-\delta)}{(1-g)+\delta} \geq 0.$$

This finally proves (16), and thus the second inequality in (15), concluding the proof of (13).

To prove (14) fix some $d \in [0, 1]$ arbitrarily and define $\mathfrak{p}(g) := g(1-2g)(-1+4g-2g^2)$, which is the radicand appearing in $F(\Delta_1(g), g, g)$. Brief calculation leads to

$$(F(d,g,g) - F(\Delta_1(g), g, g))\ \ (1-g+d) =$$
$$\underbrace{\left((4g-1)(1-g)^3 + 2g(1-2g)(1-g)d + g(1-2g)d^2\right)}_{\geq 0\ (\star)} - \underbrace{2(1-g+d)(1-g)\sqrt{\mathfrak{p}(g)}}_{\geq 0\ (\star\star)}\ .$$

To obtain $(\star)$, recall $1/3 < \beta < 1/2$ and observe that the restrictions on g, d imply $g, d \geq 0$ as well as $(4g-1) \geq 0$, $(1-g) \geq 0$, and $(1-2g) \geq 0$. Moreover we have $(1-g+d) > (1-g) \geq 0$, showing $(\star\star)$. Recall also that (10) implies that $\mathfrak{p}(g) \geq 0$ for the chosen g. Hence exploiting $(1-g+d) > 0$ and Fact 7 we obtain:

$$F(d,g,g) - F(\Delta_1(g), g, g) \geq 0$$
$$\Leftrightarrow \left((4g-1)(1-g)^3 + 2g(1-2g)(1-g)d + g(1-2g)d^2\right)^2 - 4(1-g+d)^2(1-g)^2\mathfrak{p}(g) \geq 0$$
$$\Leftrightarrow \left((1-3g)(1-g)^2 + 2g(1-2g)(1-g)d + g(1-2g)d^2\right)^2 \geq 0.$$

The last inequality is trivially true, which finally proves (14) since $(g,d) \in [1/3,\beta] \times [0,1]$ were chosen arbitrarily.

□

6 Games with more than 2 players

In this section we consider games with more than two players. A Nash equilibrium for multi-player games is defined in the same way as for two-player games. It is a choice of strategies such that no agent has a unilateral incentive to deviate. We show now how the simple 1/2-approximation algorithm for two players by Daskalakis *et al.* [6] may be generalized to a procedure that reduces the number of players in the computation of an approximate equilibrium.

Lemma 9 *Given an α-approximation algorithm for games with $k-1$ players, we can construct a $\frac{1}{2-\alpha}$-approximation algorithm for k-player games.*

Proof : Suppose we are given a game with k players. Pick any player, e.g. the first player, and fix any strategy x_1 for this player. If the first player's strategy is fixed at x_1, the game becomes a $k-1$ player game. Hence we may use the α-approximation algorithm to obtain an α-approximate equilibrium $(x_2,\ldots,x_k)$ for the players $2,...,k$ in this restricted game. Finally, player 1 computes his optimal response r_1 to $(x_2,\ldots,x_k)$ and plays a mix of his original strategy x_1 and the new strategy r_1. Let δ be the probability that player 1 plays r_1. Hence the output of this construction is $((1-\delta)x_1+\delta r_1, x_2,...,x_k)$.

We will now measure the quality of this construction. The incentive to deviate for player 1 may be bounded by $1-\delta$. For the other players the incentive may be bounded by $\alpha(1-\delta)+\delta$. By equalizing the incentives we get $\delta = \frac{1-\alpha}{2-\alpha}$, which gives the upper bound for the incentive $1-\delta = \frac{1}{2-\alpha}$. □

We may now repeatedly apply Lemma 9 combined with the 0.3393-approximation for two-player games of Spirakis and Tsaknakis [15] to get constant factor approximations for any fixed number of players. In particular, we get 0.60205-approximation for three player games and 0.71533-approximation for four-player games. To the best of our knowledge this is the first nontrivial polynomial time approximation for multiplayer normal form games.

7 Discussion

In general, our algorithms produce solutions with large support. This is to no surprise, as implied by negative results on the existence of approximate equilibrium strategies with small support [1, 9].

The major remaining open question here is whether a polynomial time algorithm for any constant $\epsilon > 0$ is possible. It would be interesting to investigate if we can exploit further the use of zero-sum games to obtain better approximations. We would also like to study if our techniques can be used for the stronger notions of approximation discussed in [12] and [8].

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