Welfare Undominated Groves Mechanisms

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Abstract. A common objective in mechanism design is to choose the outcome (for example, allocation of resources) that maximizes the sum of the agents' valuations, without introducing incentives for agents to misreport their preferences. The class of Groves mechanisms achieves this; however, these mechanisms require the agents to make payments, thereby reducing the agents' total welfare.

In this paper we introduce a measure for comparing two mechanisms with respect to the final welfare they generate. This measure induces a partial order on mechanisms and we study the question of finding minimal elements with respect to this partial order. In particular, we say a non-deficit Groves mechanism is *welfare undominated* if there exists no other non-deficit Groves mechanism that always has a smaller or equal sum of payments. We focus on two domains: (i) auctions with multiple identical units and unit-demand bidders, and (ii) mechanisms for public project problems. In the first domain we analytically characterize all welfare undominated Groves mechanisms that are anonymous and have linear payment functions, by showing that the family of optimal-in-expectation linear redistribution mechanisms, which were introduced in [6] and include the Bailey-Cavallo mechanism [1, 2], coincides with the family of welfare undominated Groves mechanisms that are anonymous and linear in the setting we study. In the second domain we show that the classic VCG (Clarke) mechanism is welfare undominated for the class of public project problems with equal participation costs, but is not undominated for a more general class.

1 Introduction

Mechanism design is often employed for coordinating group decision making among agents. Often, such mechanisms impose taxes that agents have to pay to a central authority. Although maximizing tax revenue is a desirable objective in many settings (for example, if the mechanism is an auction designed by the seller), it is not desirable in situations where no entity is profiting from the taxes. Some examples include public project problems as well as certain resource allocation problems without a seller (e.g., the right to use a shared good on a given time slot, or the exchange of take-off slots among airline companies). In such cases, we would like to have mechanisms that minimize the sum of the taxes (or, even better, achieve budget balance, that is, the sum of the taxes is zero), while maintaining other desirable properties, such as efficiency, strategy-proofness and non-deficit (i.e., the mechanism does not need to be funded by an external source).

The well-known VCG mechanism (aka. Clarke mechanism) is efficient, strategy-proof and incurs no deficit. More generally, the family of Groves mechanisms, which includes VCG, is a family of efficient and strategy-proof mechanisms. Unfortunately though, Groves mechanisms are not budget balanced. In fact, in sufficiently general settings, it is impossible to have a mechanism that satisfies efficiency, strategy-proofness, and budget balance [4].

We therefore consider the following problem: within the family of Groves mechanisms, we want to identify non-deficit mechanisms that are optimal with respect to the sum of the payments, i.e., we cannot lower the mechanism's payments without violating efficiency, strategy-proofness or the non-deficit property. Such a mechanism, in a sense, maximizes the agents' welfare (among efficient mechanisms[4]). To make this precise, we first introduce a measure for comparing two feasible mechanisms (mechanisms that are efficient, strategy-proof and satisfy the non-deficit property). We say that a feasible Groves mechanism M *welfare dominates* another feasible Groves mechanism M' if for every type vector of the agents, the sum of the payments under M is no more than the sum of the payments under M', and this holds with strict inequality for at least one type vector. This definition induces a partial order on feasible Groves mechanisms and we wish to identify minimal elements in this partial order. We call such minimal elements *welfare undominated*. Other partial orders, as well as other notions of optimality, have recently been considered in other work on redistribution mechanisms, which we discuss in Section 1.1. The notion of optimality that we study here is different from the previously studied ones at both a conceptual and a technical level, as we illustrate below.

We study the question of finding welfare undominated mechanisms in two domains. The first is auctions of multiple identical units with unit-demand bidders. In this setting, it is easy to see that VCG is welfare dominated by other Groves mechanisms, such as the Bailey-Cavallo mechanism [1, 2]. We obtain a complete characterization of linear and anonymous redistribution mechanisms that are minimal elements in this partial order: we show that a linear, anonymous Groves mechanism is welfare undominated if and only if it belongs to the class of *Optimal-in-Expectation Linear (OEL) redistribution mechanisms*, which include the Bailey-Cavallo mechanism and were introduced in [6]. The second domain is public project problems, where a set of agents must decide on financing a project (e.g., building a bridge). Here, we show that in the case where the agents have identical participation costs, no mechanism welfare dominates the VCG mechanism. On the other hand, when the participation costs can be different across agents, there exist mechanisms that welfare dominate VCG. In both domains, our proofs rely on some general properties we establish for anonymous mechanisms, which may be of independent interest (see Section 3).

1.1 Related Work

Recently, there has been a series of works on redistribution mechanisms, which are Groves mechanisms that redistribute some of the VCG payment back to the bidders. Bailey and Cavallo [1, 2] introduced a mechanism that welfare dominates VCG in some cases, such as single-item auctions, but coincides with VCG in some more general settings. We will refer to this mechanism as the BC mechanism from now on (in fact, Bailey's mechanism is not always the same as Cavallo's mechanism, but it is in the settings in which we study it). A special case of the BC mechanism was independently discovered by Porter *et al.* [14]. Cavallo also proved that the BC mechanism is optimal among the family of *surplus-anonymous* mechanisms; however, this is a quite restrictive class of mechanisms. Guo and Conitzer [8] solved for a worst-case optimal redistribution mechanism for multi-unit auctions with nonincreasing marginal values. Moulin [13] independently derived the same mechanism under a slightly different worst-case

[4] By sacrificing efficiency, it is sometimes possible to drastically lower the payments, so that the net effect is an increase in the agents' welfare [5, 3]. However, most of the prior work has focused on the case where efficiency is a hard constraint, and we will do so in this paper.

optimality notion (in the more restrictive setting of multi-unit auctions with unit demand only). These worst-case notions are different notions of optimality than the one we consider in this paper. Guo and Conitzer [6] also solve for mechanisms that maximize expected redistribution (in a certain class of mechanisms), when a prior is available. Another notion of optimality, which is closer to the one studied in this paper, was introduced in [7], namely the notion of *undominated* mechanisms. A mechanism is undominated if there is no other mechanism under which every *individual* agent pays weakly less for every type vector, and strictly less in at least one case. This is a weaker concept than ours, in the sense that for a mechanism that is undominated, there may still exist mechanisms that welfare dominate it (by increasing the payment from some agents to decrease the payments from other agents more). In the other direction, if a mechanism is welfare undominated, then it is also undominated. We believe that the notion we study in this paper is more appropriate when one is interested in the final welfare of the agents. Technically, welfare undominance appears much more challenging and seems to require different techniques.

2 Preliminaries

2.1 Tax-based mechanisms

We first briefly review tax-based mechanisms (see, e.g., [10]). Assume that there is a set of possible outcomes or ***decisions*** D, a set $\{1, \ldots, n\}$ of players where $n \geq 2$, and for each player i a set of ***types*** Θ_i and an (***initial***) ***utility function*** $v_i : D \times \Theta_i \rightarrow \mathbb{R}$. Let $\Theta := \Theta_1 \times \cdots \times \Theta_n$.

In a (direct revelation) mechanism, each player reports a type θ_i and based on this, the mechanism selects an outcome and a payment to be made by every agent. Hence a mechanism is given by a pair of functions (f, t), where f is the decision function and $t = (t_1, ..., t_n)$ is the tax function that determines the players' payments, i.e., $f : \Theta \rightarrow D$, and $t : \Theta \rightarrow \mathbb{R}^n$.

We assume that the (***final***) ***utility function*** for player i is a function $u_i : D \times \mathbb{R}^n \times \Theta_i \rightarrow \mathbb{R}$ defined by $u_i(d, t_1, \ldots, t_n, \theta_i) := v_i(d, \theta_i) + t_i$ (that is, utilities are *quasilinear*). For each vector θ of announced types, if $t_i(\theta) \geq 0$, player i ***receives*** $t_i(\theta)$, and if $t_i(\theta) < 0$, he ***pays*** $|t_i(\theta)|$. Thus when the true type of player i is θ_i and his announced type is θ'_i, his final utility is

$$u_i((f, t)(\theta'_i, \theta_{-i}), \theta_i) = v_i(f(\theta'_i, \theta_{-i}), \theta_i) + t_i(\theta'_i, \theta_{-i}),$$

where θ_{-i} are the types announced by the other players.

2.2 Properties of tax-based mechanisms

We say that a tax-based mechanism (f, t) is

- ***efficient*** if for all $\theta \in \Theta$ and $d' \in D$, $\sum_{i=1}^{n} v_i(f(\theta), \theta_i) \geq \sum_{i=1}^{n} v_i(d', \theta_i)$,
- ***budget-balanced*** if $\sum_{i=1}^{n} t_i(\theta) = 0$ for all $\theta \in \Theta$,
- ***feasible*** if $\sum_{i=1}^{n} t_i(\theta) \leq 0$ for all θ, i.e., the mechanism does not need to be funded by an external source,
- ***pay-only*** if $t_i(\theta) \leq 0$ for all θ and all $i \in \{1, \ldots, n\}$,
- ***strategy-proof*** if for all θ, $i \in \{1, \ldots, n\}$ and θ'_i,

$$u_i((f, t)(\theta_i, \theta_{-i}), \theta_i) \geq u_i((f, t)(\theta'_i, \theta_{-i}), \theta_i).$$

Tax-based mechanisms can be compared in terms of the final social welfare they generate ($\sum_{i=1}^{n} u_i((f,t)(\theta),\theta_i)$). More precisely, one can define the following two natural partial orders as a way to compare mechanisms. The first one was introduced in [7]. The second one is the concept that we introduce and study in this paper, as we believe that it is a more appropriate concept when one is interested in the final social welfare of the agents.

Definition 1. *Given two tax-based mechanisms (f,t) and (f',t') we say that (f',t')* **dominates** *(f,t) (due to [7]) if*

- *for all $\theta \in \Theta$ and all $i \in \{1,\ldots,n\}$, $u_i((f,t)(\theta),\theta_i) \le u_i((f',t')(\theta),\theta_i)$,*
- *for some $\theta \in \Theta$ and some $i \in \{1,\ldots,n\}$, $u_i((f,t)(\theta),\theta_i) < u_i((f',t')(\theta),\theta_i)$.*

Definition 2. *Given two tax-based mechanisms (f,t) and (f',t') we say that (f',t')* **welfare dominates** *(f,t) if*

- *for all $\theta \in \Theta$, $\sum_{i=1}^{n} u_i((f,t)(\theta),\theta_i) \le \sum_{i=1}^{n} u_i((f',t')(\theta),\theta_i)$,*
- *for some $\theta \in \Theta$, $\sum_{i=1}^{n} u_i((f,t)(\theta),\theta_i) < \sum_{i=1}^{n} u_i((f',t')(\theta),\theta_i)$.*

In this paper, we are interested only in Groves mechanisms, so that the decision function f is always efficient, and (welfare) dominance is strictly due to differences in the tax function t. Specifically, in this context we have that (f,t') dominates (f,t) (or simply t' dominates t) if and only if

- for all $\theta \in \Theta$ and all $i \in \{1,\ldots,n\}$, $t_i(\theta) \le t'_i(\theta)$, and
- for some $\theta \in \Theta$ and some $i \in \{1,\ldots,n\}$, $t_i(\theta) < t'_i(\theta)$,

and t' welfare dominates t if

- for all $\theta \in \Theta$, $\sum_{i=1}^{n} t_i(\theta) \le \sum_{i=1}^{n} t'_i(\theta)$, and
- for some $\theta \in \Theta$, $\sum_{i=1}^{n} t_i(\theta) < \sum_{i=1}^{n} t'_i(\theta)$.

For two tax-based mechanisms t, t', it is clear that if t' dominates t, then it also welfare dominates t. The reverse implication however does not need to hold. In Appendix A, we provide an example of two tax-based mechanisms that illustrate this.

We now define a transformation on tax-based mechanisms originating from the same decision function. This transformation was originally defined in [1] and [2] for the specific case of the VCG mechanism and in [7] for feasible Groves mechanisms. We call it the ***BCGC transformation*** after the authors of these papers.

Consider a tax-based mechanism (f,t). Given $\theta = (\theta_1,\ldots,\theta_n)$, let $T(\theta)$ be the total amount of taxes, i.e., $T(\theta) := \sum_{i=1}^{n} t_i(\theta)$. For each $i \in \{1,\ldots,n\}$ let[5]

$$S_i^{BCGC}(\theta_{-i}) := \max_{\theta'_i \in \Theta_i} T(\theta'_i, \theta_{-i}).$$

We then define the tax-based mechanism t^{BCGC} as follows:

$$t_i^{BCGC}(\theta) := t_i(\theta) - \frac{S_i^{BCGC}(\theta_{-i})}{n}.$$

The following observations generalize some of the results of [1, 2, 7].

[5] To ensure that the maximum actually exists we assume that each tax function t_i is continuous and each set of types θ_i is a compact subset of some $\mathbb{R}^k$.

Note 1.

(i) Each tax-based mechanism of the form t^{BCGC} is feasible.
(ii) If t is feasible, then either t and t^{BCGC} coincide or t^{BCGC} dominates t.

Proof. *(i)* For all θ we have $T(\theta) \le S_i^{BCGC}(\theta_{-i})$, so

$$T^{BCGC}(\theta) = \sum_{i=1}^{n} t_i^{BCGC}(\theta) = T(\theta) - \sum_{i=1}^{n} \frac{S_i^{BCGC}(\theta_{-i})}{n}$$
$$= \sum_{i=1}^{n} \frac{T(\theta) - S_i^{BCGC}(\theta_{-i})}{n} \le 0.$$

(ii) If t is feasible, then for all θ and all $i \in \{1, \ldots, n\}$ we have $S_i^{BCGC}(\theta_{-i}) \le 0$, and hence $t_i^{BCGC}(\theta) \ge t_i(\theta)$. □

2.3 Groves mechanisms

Each ***Groves mechanism*** is a tax-based mechanism (f, t) such that the following hold[6]:

- $f(\theta) \in \arg\max_d \sum_{i=1}^{n} v_i(d, \theta_i)$, i.e., the chosen outcome maximizes the initial social welfare.
- $t_i : \Theta \to \mathbb{R}$ is defined by $t_i(\theta) := g_i(\theta) + h_i(\theta_{-i})$,
- $g_i(\theta) := \sum_{j \ne i} v_j(f(\theta), \theta_j)$,
- $h_i : \Theta_{-i} \to \mathbb{R}$ is an arbitrary function.

Intuitively, $g_i(\theta)$ represents the (initial) social welfare from the decision $f(\theta)$, when player i's (initial) utility is not counted. Recall now the following crucial result, see e.g., [10].

Groves Theorem Every Groves mechanism (f, t), is efficient and strategy-proof.

For several decision problems the only efficient and strategy-proof tax-based mechanisms are Groves mechanisms. By a general result of [9] this is the case for both domains that we consider in this paper and explains our focus on Groves mechanisms.

A feasible Groves mechanism is ***undominated*** if there is no other feasible Groves mechanism that dominates it [7]. A feasible Groves mechanism is ***welfare undominated*** if there is no other feasible Groves mechanism that welfare dominates it. Welfare undominance is a strictly stronger concept than undominance, as is illustrated in Appendix A.

A special Groves mechanism, called the ***VCG*** or ***Clarke*** mechanism, is obtained using[7]

$$h_i(\theta_{-i}) := -\max_{d \in D} \sum_{j \ne i} v_j(d, \theta_j).$$

In this case,

$$t_i(\theta) := \sum_{j \ne i} v_j(f(\theta), \theta_j) - \max_{d \in D} \sum_{j \ne i} v_j(d, \theta_j),$$

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[6] Here and below $\sum_{j \ne i}$ is a shorthand for the summation over all $j \in \{1, \ldots, n\}$, $j \ne i$.

[7] Here and below, to ensure that the considered maximum exist, we assume that f and each v_i are continuous functions and D and each θ_i are compact subsets of some $\mathbb{R}^k$.

which shows that the VCG mechanism is pay-only.

Following [2], let us now consider the mechanism that results from applying the BCGC transformation to the VCG mechanism. We refer to this as the Bailey-Cavallo mechanism or simply the BC mechanism. Let $\theta' := (\theta_1, ..., \theta_{i-1}, \theta'_i, \theta_{i+1}, ..., \theta_n)$, so $\theta'_j = \theta_j$ for $j \neq i$ and the ith player's type in the type vector θ' is θ'_i. Then

$$S_i^{BCGC}(\theta_{-i}) = \max_{\theta'_i \in \Theta_i} \sum_{k=1}^{n} \left[\sum_{j \neq k} v_j(f(\theta'), \theta'_j) - \max_{d \in D} \sum_{j \neq k} v_j(d, \theta'_j) \right],$$

that is,

$$S_i^{BCGC}(\theta_{-i}) = \max_{\theta'_i \in \Theta_i} \left[(n-1) \sum_{k=1}^{n} v_k(f(\theta'), \theta'_k) - \sum_{k=1}^{n} \max_{d \in D} \sum_{j \neq k} v_j(d, \theta'_j) \right]. \quad (1)$$

In many settings, we have that for all θ and for all i, $S_i^{BCGC}(\theta_{-i}) = 0$, and consequently the VCG and BC mechanisms coincide. Whenever they do not, by Note 1(ii) BC dominates VCG. This is the case for the single-item auction, as it can be seen that there $S_i^{BCGC}(\theta_{-i}) = -[\theta_{-i}]_2$, where $[\theta_{-i}]_2$ is the second-highest bid among bids other than player i's own bid.

3 Anonymous Groves mechanisms

Throughout this paper, we will be interested in a special class of Groves mechanisms that are called anonymous Groves mechanisms. We provide here some results about this class that we will utilize in later sections. We call a function $f : A^n \to B$ ***permutation independent*** if for all permutations π of $\{1, \ldots, n\}$, $f = f \circ \pi$. Following [12] we call a Groves mechanism (determined by the vector of functions) $(h_1, \ldots, h_n)$ ***anonymous*** if

- all type sets Θ_i are equal,
- all functions h_i coincide and each of them is permutation independent.

Hence, an anonymous Groves mechanism is uniquely determined by a single function $h : \Theta^{n-1} \to \mathbb{R}$.

In general, the VCG mechanism is not anonymous. But it is anonymous when all the type sets are equal and all the initial utility functions v_i coincide. This is the case in both of the domains that we consider in this paper.

For any $\theta \in \Theta$ and any permutation π of $\{1, \ldots, n\}$ we define $\theta^\pi \in \Theta$ by letting

$$\theta_i^\pi := \theta_{\pi^{-1}(i)}.$$

Denote by $\Pi(k)$ the set of all permutations of the set $\{1, \ldots, k\}$. Given a Groves mechanism $h := (h_1, \ldots, h_n)$ for which the type set Θ_i is the same for every player (and equal to, say, Θ_0) we construct now a function $h' : \Theta_0^{n-1} \to \mathbb{R}$ by putting

$$h'(x) := \frac{\sum_{\pi \in \Pi(n-1)} \sum_{j=1}^{n} h_j(x^\pi)}{n!},$$

where x^π is defined analogously to θ^π.

Note that h' is permutation independent, so h' is an anonymous Groves mechanism.

The following lemma shows that some of the properties of h transfer to h'.

Lemma 1. *Consider a Groves mechanism h and the corresponding anonymous Groves mechanism h'. Let $G(\theta) := \sum_{j=1}^{n} v_j(f(\theta), \theta_j)$. Suppose that for all permutations π of $\{1, \ldots, n\}$, $G(\theta) = G(\theta^\pi)$. Then:*

(i) If h is feasible, so is h'.
(ii) If an anonymous Groves mechanism h^0 is welfare dominated by h, then it is welfare dominated by h'.

The proof can be found in Appendix B. The assumption in Lemma 1 of permutation independence of $G(\cdot)$ is satisfied in both of the domains that we consider in this paper. Basically, Lemma 1 says that if a Groves mechanism is not welfare undominated, then it must be welfare dominated by an anonymous Groves mechanism.

4 Multi-unit auctions with unit demand

In this section, we consider auctions where there are multiple identical units of a single good and all players have unit demand, i.e., each player wants only one unit. (When there is only one unit, we have a standard single-item auction.) For this setting, we obtain an analytical characterization of all welfare undominated Groves mechanisms that are anonymous and have linear payment functions, by proving that the optimal-in-expectation linear redistribution mechanisms (OEL mechanisms) [6], which include the BC mechanism, are the only welfare undominated Groves mechanisms that are anonymous and linear. We also show that undominance and welfare undominance are equivalent if we restrict our consideration to Groves mechanisms that are anonymous and linear in the setting of multi-unit auctions with unit demand.

4.1 Optimal-in-expectation linear redistribution mechanisms

The optimal-in-expectation linear redistribution mechanisms are special cases of Groves mechanisms that are anonymous and linear. The OEL mechanisms are defined only for multi-unit auctions with unit demand. In a unit demand multi-unit auction, there are m indistinguishable units for sale, and each player is interested in only one unit. For player i, her type θ_i is her valuation for winning one unit. We assume all bids (announced types) are bounded below by L and above by U, i.e., $\Theta_i = [L, U]$. (We note that L can be 0.)

The tax function t of an anonymous linear Groves mechanism is defined as $t_i(\theta) = t_i^{VCG}(\theta) + r(\theta_{-i})$ for all i and θ. Here t^{VCG} is (the tax function of) the VCG mechanism, and r is a linear function defined as $r(\theta_{-i}) = c_0 + \sum_{j=1}^{n-1} c_j[\theta_{-i}]_j$ (where $[\theta_{-i}]_j$ is the jth highest bid among θ_{-i}). For OEL, the c_j's are chosen according to one of the following options (indexed by k, k is from 0 to n, and $k - m$ is odd):

$\mathbf{k = 0}$:
$c_i = (-1)^{m-i}\binom{n-i-1}{n-m-1}/\binom{m-1}{i-1}$ for $i = 1, \ldots, m$,
$c_0 = Um/n - U\sum_{i=1}^{m}(-1)^{m-i}\binom{n-i-1}{n-m-1}/\binom{m-1}{i-1}$, and $c_i = 0$ for other i.

$\mathbf{k = 1, 2, \ldots, m}$:
$c_i = (-1)^{m-i}\binom{n-i-1}{n-m-1}/\binom{m-1}{i-1}$ for $i = k+1, \ldots, m$,

$c_k = m/n - \sum_{i=k+1}^{m}(-1)^{m-i}\binom{n-i-1}{n-m-1}/\binom{m-1}{i-1}$, and $c_i = 0$ for other i.

$\mathbf{k = m+1, m+2, \ldots, n-1}$:
$c_i = (-1)^{m-i-1}\binom{i-1}{m-1}/\binom{n-m-1}{n-i-1}$ for $i = m+1, \ldots, k-1$,
$c_k = m/n - \sum_{i=m+1}^{k-1}(-1)^{m-i-1}\binom{i-1}{m-1}/\binom{n-m-1}{n-i-1}$, and $c_i = 0$ for other i.

$\mathbf{k = n}$:
$c_i = (-1)^{m-i-1}\binom{i-1}{m-1}/\binom{n-m-1}{n-i-1}$ for $i = m+1, \ldots, n-1$,
$c_0 = Lm/n - L\sum_{i=m+1}^{n-1}(-1)^{m-i-1}\binom{i-1}{m-1}/\binom{n-m-1}{n-i-1}$, and $c_i = 0$ for other i.

For example, when $k = m+1$, we have $c_{m+1} = m/n$ and $c_i = 0$ for all other i. For this specific OEL mechanism, $t_i^{OEL}(\theta) = t_i^{VCG}(\theta) + \frac{m}{n}[\theta_{-i}]_{m+1}$. That is, besides paying the VCG payment, every player receives an amount that is equal to m/n times the $(m+1)$th highest bid from the other players. Actually, this is the BC mechanism for this setting.

One property of the OEL mechanisms is that the sum of the taxes $\sum_{i=1}^{n} t_i^{OEL}(\theta)$ is always less than or equal to 0 and it equals 0 whenever

- $[\theta]_1 = U$, if $k = 0$.
- $[\theta]_{k+1} = [\theta]_k$, if $k \in \{1, \ldots, n-1\}$.
- $[\theta]_n = L$, if $k = n$.

Using this property, we will prove that the OEL mechanisms are the only welfare undominated Groves mechanisms that are anonymous and linear.

4.2 Characterization of welfare undominated Groves mechanisms that are anonymous and linear

We first show that the OEL mechanisms are welfare undominated. (It has previously been shown that they are undominated [7], but as we pointed out, being welfare undominated is a stronger property.)

Theorem 1. *There is no feasible Groves mechanism that welfare dominates an OEL mechanism.*

According to Lemma 1, we only need to prove this for the case of anonymous Groves mechanisms:

Lemma 2. *There is no feasible anonymous Groves mechanism that welfare dominates an OEL mechanism.*

Proof. We first prove: *no OEL mechanism with index $k \in \{1, \ldots, n-1\}$ is welfare dominated by a feasible anonymous Groves mechanism.*

Suppose a feasible anonymous Groves mechanism (corresponding to the tax function) t welfare dominates an OEL mechanism (corresponding to the tax function) t^{OEL} with index $k \in \{1, \ldots, n-1\}$.

Both t and t^{OEL} are tax functions of anonymous Groves mechanisms. For any i and any θ, we can write $t_i(\theta)$ as $t_i^{VCG}(\theta) + h(\theta_{-i})$, and we can write $t_i^{OEL}(\theta)$ as $t_i^{VCG}(\theta) + h^{OEL}(\theta_{-i})$. For any i and θ_{-i}, we define the following function: $\Delta(\theta_{-i}) = h(\theta_{-i}) - h^{OEL}(\theta_{-i})$.

Since t welfare dominates t^{OEL}, we have that for any θ, $\sum_{i=1}^{n} t_i(\theta) \geq \sum_{i=1}^{n} t_i^{OEL}(\theta)$. That is, for any θ, $\sum_{i=1}^{n} \Delta(\theta_{-i}) \geq 0$.

We also have that, whenever $[\theta]_{k+1} = [\theta]_k$, we have $\sum_{i=1}^{n} t_i^{OEL}(\theta) = 0$; in this case, because t is feasible, we must have $\sum_{i=1}^{n} t_i(\theta) = 0$ and hence $\sum_{i=1}^{n} \Delta(\theta_{-i}) = 0$.

Now we claim that $\Delta(\theta_{-i}) = 0$ for all θ_{-i}.

Let $c(\theta_{-i})$ be the number of bids among θ_{-i} that equal $[\theta_{-i}]_k$. Hence, we must show that for all θ_{-i} with $c(\theta_{-i}) \geq 1$, we have $\Delta(\theta_{-i}) = 0$.

We now prove it by induction on the value of c (backwards, from $n-1$ to 1).

Base case: $c = n - 1$.

Suppose there is a θ_{-i} with $c(\theta_{-i}) = n-1$. That is, all the bids in θ_{-i} are identical. When θ_i is also equal to the bids in θ_{-i}, all bids in θ are the same so that $[\theta]_{k+1} = [\theta]_k$. Hence, by our earlier observation, we have $\sum_{j=1}^{n} \Delta(\theta_{-j}) = 0$. But we know that for all j, $\Delta(\theta_{-j})$ is the same value. Hence $\Delta(\theta_{-i}) = 0$ for all θ_{-i} when $c(\theta_{-i}) = n-1$.

Induction step.

Let us assume that for all θ_{-i}, if $c(\theta_{-i}) \geq p$ (where $p \in \{2, \ldots, n-1\}$), then $\Delta(\theta_{-i}) = 0$. Now we consider any θ_{-i} with $c(\theta_{-i}) = p-1$. When θ_i is equal to $[\theta_{-i}]_k$, we have $[\theta]_k = [\theta]_{k+1}$, which implies that $\sum_{j=1}^{n} \Delta(\theta_{-j}) = 0$. For all j with $\theta_j = [\theta_{-i}]_k$, $\Delta(\theta_{-j}) = \Delta(\theta_{-i})$, and for other j, $c(\theta_{-j}) = p$. Therefore, by the induction assumption, $\sum_{j=1}^{n} \Delta(\theta_{-j})$ is a positive multiple of $\Delta(\theta_{-i})$, which implies that $\Delta(\theta_{-i}) = 0$.

By induction, we have shown that $\Delta(\theta_{-i}) = 0$ for all θ_{-i}. This implies that t and t^{OEL} are identical. Hence, no other feasible anonymous Groves mechanism welfare dominates an OEL mechanism with index $k \in \{1, \ldots, n-1\}$.

Now we prove: *no OEL mechanism with index $k = 0$ is welfare dominated by a feasible anonymous Groves mechanism.*

Suppose a feasible anonymous Groves mechanism (corresponding to the tax function) t welfare dominates an OEL mechanism (corresponding to the tax function) t^{OEL} with index $k = 0$.

Both t and t^{OEL} are tax functions of anonymous Groves mechanisms. For any i and any θ, we can write $t_i(\theta)$ as $t_i^{VCG}(\theta) + h(\theta_{-i})$, and we can write $t_i^{OEL}(\theta)$ as $t_i^{VCG}(\theta) + h^{OEL}(\theta_{-i})$. For any i and θ_{-i}, we define the following function: $\Delta(\theta_{-i}) = h(\theta_{-i}) - h^{OEL}(\theta_{-i})$.

Since t welfare dominates t^{OEL}, we have that for any θ, $\sum_{i=1}^{n} t_i(\theta) \geq \sum_{i=1}^{n} t_i^{OEL}(\theta)$. That is, for any θ, $\sum_{i=1}^{n} \Delta(\theta_{-i}) \geq 0$.

We also have that, whenever $[\theta]_1 = U$, we have $\sum_{i=1}^{n} t_i^{OEL}(\theta) = 0$; in this case, because t is feasible, we must have $\sum_{i=1}^{n} t_i(\theta) = 0$ and hence $\sum_{i=1}^{n} \Delta(\theta_{-i}) = 0$.

Now we claim that $\Delta(\theta_{-i}) = 0$ for all θ_{-i}.

Let $c(\theta_{-i})$ be the number of bids among θ_{-i} that equal U. Hence, we must show that for all θ_{-i} with $c(\theta_{-i}) \geq 0$, we have $\Delta(\theta_{-i}) = 0$.

We now prove it by induction on the value of c (backwards, from $n-1$ to 0).

Base case: $c = n - 1$.

Suppose there is a θ_{-i} with $c(\theta_{-i}) = n-1$. That is, all the bids in θ_{-i} are equal to U. When θ_i is also equal to the bids in U, by our earlier observation, we have $\sum_{j=1}^{n} \Delta(\theta_{-j}) = 0$. But we know that for all j, $\Delta(\theta_{-j})$ is the same value. Hence $\Delta(\theta_{-i}) = 0$ for all θ_{-i} when $c(\theta_{-i}) = n-1$.

Induction step.

Let us assume that for all θ_{-i}, if $c(\theta_{-i}) \geq p$ (where $p \in \{2, \ldots, n-1\}$), then $\Delta(\theta_{-i}) = 0$. Now we consider any θ_{-i} with $c(\theta_{-i}) = p - 1$. When θ_i is equal to U, we have $[\theta]_1 = U$, which implies that $\sum_{j=1}^{n} \Delta(\theta_{-j}) = 0$. For all j with $\theta_j = U$, $\Delta(\theta_{-j}) = \Delta(\theta_{-i})$, and for other j, $c(\theta_{-j}) = p$. Therefore, by the induction assumption, $\sum_{j=1}^{n} \Delta(\theta_{-j})$ is a positive multiple of $\Delta(\theta_{-i})$, which implies that $\Delta(\theta_{-i}) = 0$.

By induction, we have shown that $\Delta(\theta_{-i}) = 0$ for all θ_{-i}. This implies that t and t^{OEL} are identical. Hence, no other feasible anonymous Groves mechanism welfare dominates an OEL mechanism with index $k = 0$.

It remains to prove: *no OEL mechanism with index $k = n$ is welfare dominated by a feasible anonymous Groves mechanism.*

This case is similar to the case of $k = 0$ and we omit it here.

We now show that within the family of anonymous and linear Groves mechanisms, the OEL mechanisms are the only ones that are welfare undominated. Actually, they are also the only ones that are undominated, which is a stronger claim since being undominated is a weaker property.

Theorem 2. *If a feasible anonymous linear Groves mechanism is undominated, then it must be an OEL mechanism.*

The proof of this theorem is in Appendix C. Hence, we have the following complete characterization in this context:

Corollary 1. *A feasible anonymous linear Groves mechanism is (welfare) undominated if and only if it is an OEL mechanism.*

The above corollary also shows that if we consider only Groves mechanisms that are anonymous and linear in the setting of multi-unit auctions with unit demand, then undominance and welfare undominance are equivalent.[8]

5 Public project problem with equal participation costs

We now study a well known class of decision problems, namely public project problems—see, e.g., [10, 12, 11].

Public project problem
Consider $(D, \Theta_1, \ldots, \Theta_n, v_1, \ldots, v_n)$, where

- $D = \{0, 1\}$ (reflecting whether a project is canceled or takes place),
- for all $i \in \{1, \ldots, n\}$, $\Theta_i = [0, c]$, where $c > 0$,
- for all $i \in \{1, \ldots, n\}$, $v_i(d, \theta_i) := d(\theta_i - \frac{c}{n})$,

[8] Thus, we have also characterized all undominated Groves mechanisms that are anonymous and linear. There is no corresponding result in [7].

In this setting a set of n agents needs to decide on financing a project of cost c. In the case that the project takes place, each agent contributes the same share, c/n, so as to cover the total cost. Hence the participation costs of all players are the same. When the players employ a tax-based mechanism to decide on the project, then in addition to c/n, each player also has to pay or receive the tax, $t_i(\theta)$, imposed by the mechanism.

By the result of Holmstrom [9], the only efficient and strategy-proof tax-based mechanisms in this domain are Groves mechanisms. To determine the efficient outcome for a given type vector θ, note that $\sum_{i=1}^{n} v_i(d, \theta_i) = d(\sum_{i=1}^{n} \theta_i - c)$. Hence efficiency here for a mechanism (f, t) means that $f(\theta) = 1$ if $\sum_{i=1}^{n} \theta_i \geq c$ and $f(\theta) = 0$ otherwise, i.e., the project takes place if and only if the declared total value that the agents have for the project exceeds its cost.

We first observe the following result.

Note 2. In the public project problem the BC mechanism coincides with VCG.

Proof. It suffices to check that in equation (1) it holds that $S_i^{BCGC}(\theta_{-i}) = 0$ for all i and all θ_{-i}. By the feasibility of VCG we have $S_i^{BCGC} \leq 0$, hence all we need is to show that there is a value for θ'_i that makes the expression in (1) equal to 0. Checking this is quite simple. If $\sum_{j \neq i} \theta_j < \frac{n-1}{n} c$, then we take $\theta'_i := 0$ and otherwise $\theta'_i := c$. □

We now show that in fact VCG cannot be improved upon. Before stating our result, we would like to note that one ideally would like to have a mechanism that is budget-balanced, i.e., $\sum_i t_i(\theta) = 0$ for all θ, so that in total the agents only pay the cost of the project and no more. However this is not possible and as explained in [10, page 861-862], for the public project problem no mechanism exists that is efficient, strategy-proof and budget balanced. Our theorem below considerably strengthens this result, showing that VCG is optimal with respect to minimizing the total payment of the players.

Theorem 3. *In the public project problem there exists no feasible Groves mechanism that welfare dominates the VCG mechanism.*

As in Section 4, we first establish the desired conclusion for anonymous Groves mechanisms and then extend it to arbitrary ones by Lemma 1.

Lemma 3. *In the public project problem there exists no anonymous feasible Groves mechanism that welfare dominates the VCG mechanism.*

The proof can be found in Appendix D.

6 Public project problem: the general case

The assumption that we have made so far in the public project problem that each player's cost share is the same may not always be realistic. Indeed, it may be argued that 'richer' players (read: larger enterprises) should contribute more. Does it matter if we modify the formulation of the problem appropriately? The answer is 'yes'. First, let us formalize this problem. We assume now that each (initial) utility function is of the form

$$v_i(d, \theta_i) := d(\theta_i - c_i),$$

where for all $i \in \{1, \ldots, n\}$, $c_i > 0$ and $\sum_{i=1}^{n} c_i = c$.

In this setting, c_i is the cost share of the project cost to be financed by player i. We call the resulting problem the ***general public project problem***. It is taken from [11, page 518].

We begin by proving the following optimality result concerning the VCG mechanism.

Theorem 4. *In the general public project problem there is no pay-only Groves mechanism that dominates the VCG mechanism.*

The proof can be found in Appendix E.

It remains an open problem whether the above result can be extended to the welfare dominance relation. On the other hand, the above theorem cannot be extended to feasible Groves mechanisms, as the following result holds.

Theorem 5. *For any $n \geq 3$, an instance of the general public project problem with n players exists for which the BC mechanism dominates the VCG mechanism.*

The proof can be found in Appendix F.

By virtue of Theorem 4, the BC mechanism in the proof of the above theorem is not pay-only.

7 Summary

In this paper, we introduced and studied the following relation on feasible Groves mechanisms: a feasible Groves mechanism *welfare dominates* another feasible Groves mechanism if the total welfare (with taxes taken into account) under the former is at least as great as the total welfare under the latter, for any type vector—and the inequality is strict for at least one type vector. This dominance notion is different from the one proposed in [7], as we illustrate in Appendix A. We then studied welfare (un)dominance in two domains. The first domain we considered was that of auctions with multiple identical units and unit demand bidders. In this domain, we analytically characterized all welfare undominated Groves mechanisms that are anonymous and have linear payment functions. The second domain we considered is that of public project problems. In this domain, we showed that the VCG mechanism is welfare undominated if cost shares are equal, but also that this is not necessarily true if cost shares are not necessarily equal (though we showed that the VCG mechanism remains undominated in the weaker sense of [7] among pay-only mechanisms in this more general setting).

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A Dominance is distinct from welfare dominance

In this appendix, we give two tax-based mechanisms t and t' such that t' welfare dominates t, but t' does not dominate t. Both t and t' are feasible anonymous Groves mechanisms.

Consider a single-item auction with 4 players. We assume that for each player, the set of allowed bids is the same, namely, integers from 0 to 3.

Let t^{VCG} be (the tax function of) the VCG mechanism. For all $\theta \in \{0,1,2,3\}^4$, $\sum_{i=1}^{4} t_i^{VCG}(\theta) = -[\theta]_2$. This is because for a single-item auction, the VCG mechanism is the second-price auction.

We define t and t' as follows:

For all θ, $t_i(\theta) := t_i^{VCG}(\theta) + h(\theta_{-i})$, where $h(\theta_{-i}) = r([\theta_{-i}]_1, [\theta_{-i}]_2, [\theta_{-i}]_3)$, and the function r is given in the table below. (We recall that $[\theta_{-i}]_j$ is the jth-highest bid among bids other than i's own bid.)

For all θ, $t'_i(\theta) := t_i^{VCG}(\theta) + h'(\theta_{-i})$, where $h'(\theta_{-i}) = r'([\theta_{-i}]_1, [\theta_{-i}]_2, [\theta_{-i}]_3)$, and the function r' is given in the table below.

$r(0,0,0)$	0	$r'(0,0,0)$	0
$r(1,0,0)$	0	$r'(1,0,0)$	0
$r(1,1,0)$	1/4	$r'(1,1,0)$	1/4
$r(1,1,1)$	1/4	$r'(1,1,1)$	1/4
$r(2,0,0)$	0	$r'(2,0,0)$	0
$r(2,1,0)$	1/12	$r'(2,1,0)$	7/24
$r(2,1,1)$	0	$r'(2,1,1)$	1/6
$r(2,2,0)$	1/2	$r'(2,2,0)$	1/2
$r(2,2,1)$	0	$r'(2,2,1)$	1/4
$r(2,2,2)$	1/2	$r'(2,2,2)$	1/2
$r(3,0,0)$	0	$r'(3,0,0)$	0
$r(3,1,0)$	1/4	$r'(3,1,0)$	1/4
$r(3,1,1)$	0	$r'(3,1,1)$	1/4
$r(3,2,0)$	2/3	$r'(3,2,0)$	2/3
$r(3,2,1)$	1	$r'(3,2,1)$	19/24
$r(3,2,2)$	0	$r'(3,2,2)$	1/6
$r(3,3,0)$	2/3	$r'(3,3,0)$	5/6
$r(3,3,1)$	0	$r'(3,3,1)$	7/12
$r(3,3,2)$	1	$r'(3,3,2)$	5/6
$r(3,3,3)$	0	$r'(3,3,3)$	1/2

With the above characterization, t' welfare dominates t (the total tax under t' is never lower, and in some cases it is strictly higher: for example, for the bid vector $(3,2,2,2)$, the sum of the r_i is $1/2$, but the sum of the r'_i is 1). On the other hand, t' does not dominate t: for example, $r(3,3,2) = 1 > 5/6 = r'(3,3,2)$. In fact, no feasible Groves mechanism dominates t.

B Proof of Lemma 1

For all $\theta \in \Theta$ we have

$$\sum_{i=1}^{n} h'_i(\theta_{-i}) = \frac{\sum_{i=1}^{n} \sum_{\pi \in \Pi(n-1)} \sum_{j=1}^{n} h_j((\theta_{-i})^{\pi})}{n!} =$$

$$\frac{\sum_{\pi \in \Pi(n)} \sum_{i=1}^{n} h_i(\theta^{\pi}_{-i})}{n!}$$

where the last equality holds since in both terms we aggregate over all applications of all h_i functions to all permutations of $n-1$ elements of θ.

Let t and t' be the tax functions of the mechanisms h and h', respectively. We have

$$\sum_{i=1}^{n} t'_i(\theta) = (n-1)G(\theta) + \sum_{i=1}^{n} h'_i(\theta_{-i})$$

and for all $\pi \in \Pi(n)$

$$\sum_{i=1}^{n} t_i(\theta^{\pi}) = (n-1)G(\theta^{\pi}) + \sum_{i=1}^{n} h_i(\theta^{\pi}_{-i}).$$

Hence by the assumption about $G(\cdot)$

$$\sum_{i=1}^{n} t'_i(\theta) = \frac{\sum_{\pi \in \Pi(n)} \sum_{i=1}^{n} t_i(\theta^{\pi})}{n!} \tag{2}$$

(i) is now an immediate consequence of (2).

To prove (ii) let t^0 be the tax function of h^0. h welfare dominates h^0, so for all $\theta \in \Theta$ and all $\pi \in \Pi(n)$

$$\sum_{i=1}^{n} t_i(\theta^{\pi}) \geq \sum_{i=1}^{n} t^0_i(\theta^{\pi})$$

with at least one inequality strict. Hence for all $\theta \in \Theta$

$$\frac{\sum_{\pi \in \Pi(n)} \sum_{i=1}^{n} t_i(\theta^{\pi})}{n!} \geq \frac{\sum_{\pi \in \Pi(n)} \sum_{i=1}^{n} t^0_i(\theta^{\pi})}{n!}$$

with at least one inequality strict.

But the fact that h^0 is anonymous and the assumption about $G(\cdot)$ imply that for all $\theta \in \Theta$ and all permutations π of $\{1, \ldots, n\}$

$$\sum_{i=1}^{n} t^0_i(\theta^{\pi}) = \sum_{i=1}^{n} t^0_i(\theta),$$

so by (2) and the above inequality all $\theta \in \Theta$

$$\sum_{i=1}^{n} t'_i(\theta) \geq \sum_{i=1}^{n} t^0_i(\theta),$$

with at least one inequality strict.

C Proof of Theorem 2

Before proving this theorem, let us introduce the following lemma:

Lemma 4. *Let I be the set of points $(s_1, s_2, \ldots, s_k)$ $(U \geq s_1 \geq s_2 \geq \ldots \geq s_k \geq L)$ that satisfy $Q_0 + Q_1 s_1 + Q_2 s_2 + \ldots + Q_k s_k = 0$ (the Q_i are constants). If the measure of I is positive (Lebesgue measure on R^k), then $Q_i = 0$ for all i.*

Proof. If $Q_i \neq 0$ for some i, then for any $U \geq s_1 \geq s_2 \geq \ldots \geq s_{i-1} \geq s_{i+1} \geq \ldots \geq s_k \geq L$, to make $Q_0 + Q_1 s_1 + Q_2 s_2 + \ldots + Q_k s_k = 0$, s_i can take at most one value. As a result the measure of I must be 0.

Proof of Theorem 2. Let t be a feasible anonymous linear Groves mechanism. We recall that a Groves mechanism is anonymous and linear if the tax function is defined as $t_i(\theta) = t_i^{VCG}(\theta) + r(\theta_{-i})$, and r is a linear function defined as $r(\theta_{-i}) = a_0 + \sum_{j=1}^{n-1} a_j [\theta_{-i}]_j$ ($[\theta_{-i}]_j$ is the jth highest type among θ_{-i}, and the a_j are constants).

For any θ, the total payment $\sum_{i=1}^{n} t_i(\theta) = \sum_{i=1}^{n} t_i^{VCG}(\theta) + na_0 + \sum_{i=1}^{n} \sum_{j=1}^{n-1} a_j [\theta_{-i}]_j$. In our setting, the total VCG payment $\sum_{i=1}^{n} t_i^{VCG}(\theta) = -m[\theta]_{m+1}$, which is a linear function in terms of the types among θ. Therefore, for any θ, the total payment $\sum_{i=1}^{n} t_i(\theta)$ is a linear function in terms of the types among θ. For simplicity, we rewrite the total payment as $C_0 + C_1[\theta]_1 + C_2[\theta]_2 + \ldots + C_n[\theta]_n$. The C_i are constants determined by the a_i. We have

$$\begin{gathered} C_0 = na_0 \\ C_1 = (n-1)a_1 \\ C_2 = a_1 + (n-2)a_2 \\ C_3 = 2a_2 + (n-3)a_3 \\ \vdots \\ C_m = (m-1)a_{m-1} + (n-m)a_m \\ C_{m+1} = ma_m + (n-m-1)a_{m+1} - m \\ C_{m+2} = (m+1)a_{m+1} + (n-m-2)a_{m+2} \\ \vdots \\ C_{n-1} = (n-2)a_{n-2} + a_{n-1} \\ C_n = (n-1)a_{n-1} \end{gathered}$$

Given any multiset of other types θ_{-i}, for any possible value of θ_i, we must have $\sum_{i=1}^{n} t_i(\theta) \leq 0$ (feasibility). That is, for any θ_{-i}, we have $\max_{\theta_i} \sum_{i=1}^{n} t_i(\theta) \leq 0$. If for some θ_{-i}, we have $\max_{\theta_i} \sum_{i=1}^{n} t_i(\theta) < -\epsilon$ ($\epsilon > 0$), then we can increase the payment of agent i by ϵ without violating feasibility, when the other agents' types are θ_{-i}. Therefore, if mechanism t is undominated, for any θ_{-i}, we have $\max_{\theta_i} \sum_{i=1}^{n} t_i(\theta) = 0$.

We denote $[\theta_{-i}]_j$ by s_j ($j = 1, \ldots, n-1$). That is, $s_1 \geq s_2 \geq \ldots \geq s_{n-1}$.

$\max_{\theta_i} \sum_{i=1}^{n} t_i(\theta)$ then equals the maximum of the following expressions:

$$\max_{L \leq \theta_i \leq s_{n-1}} \sum_{i=1}^{n} t_i(\theta)$$
$$\max_{s_{n-1} \leq \theta_i \leq s_{n-2}} \sum_{i=1}^{n} t_i(\theta)$$
$$\vdots$$
$$\max_{s_2 \leq \theta_i \leq s_1} \sum_{i=1}^{n} t_i(\theta)$$
$$\max_{s_1 \leq \theta_i \leq U} \sum_{i=1}^{n} t_i(\theta)$$

We take a closer look at $\max_{L \leq \theta_i \leq s_{n-1}} \sum_{i=1}^{n} t_i(\theta)$. When $L \leq \theta_i \leq s_{n-1}$, the jth highest type $[\theta]_j = s_j$ for $j = 1, \ldots, n-1$, and the nth highest type $[\theta]_n = \theta_i$ (this case corresponds to agent i being the agent with the lowest type). We have $\max_{L \leq \theta_i \leq s_{n-1}} \sum_{i=1}^{n} t_i(\theta) = \max_{L \leq \theta_i \leq s_{n-1}} (C_0 + C_1 s_1 + C_2 s_2 + \ldots + C_{n-1} s_{n-1} + C_n \theta_i) = \max\{C_0 + C_1 s_1 + \ldots + C_{n-1} s_{n-1} + C_n L, C_0 + C_1 s_1 + \ldots + C_{n-1} s_{n-1} + C_n s_{n-1}\}$. That is, because the expression is linear, the maximum is reached when θ_i is set to either the lower bound L or the upper bound s_{n-1}.

Similarly, we have $\max_{s_{n-1} \leq \theta_i \leq s_{n-2}} \sum_{i=1}^{n} t_i(\theta) = \max\{C_0 + C_1 s_1 + C_2 s_2 + \ldots + C_{n-2} s_{n-2} + C_{n-1} s_{n-1} + C_n s_{n-1}, C_0 + C_1 s_1 + C_2 s_2 + \ldots + C_{n-2} s_{n-2} + C_{n-1} s_{n-2} + C_n s_{n-1}\}$.

$\vdots$

$\max_{s_2 \leq \theta_i \leq s_1} \sum_{i=1}^{n} t_i(\theta) = \max\{C_0 + C_1 s_1 + C_2 s_1 + C_3 s_2 + \ldots + C_n s_{n-1}, C_0 + C_1 s_1 + C_2 s_2 + C_3 s_2 + \ldots + C_n s_{n-1}\}$.

$\max_{s_1 \leq \theta_i \leq U} \sum_{i=1}^{n} t_i(\theta) = \max\{C_0 + C_1 U + C_2 s_1 + \ldots + C_n s_{n-1}, C_0 + C_1 s_1 + C_2 s_1 + \ldots + C_n s_{n-1}\}$.

Putting all the above together, we have that for any $U \geq s_1 \geq s_2 \geq \ldots \geq s_{n-1} \geq L$, the maximum of the following expressions is 0.

- (n): $C_0 + C_1 s_1 + C_2 s_2 + \ldots + C_{n-1} s_{n-1} + C_n L$
- ($n-1$): $C_0 + C_1 s_1 + C_2 s_2 + \ldots + C_{n-1} s_{n-1} + C_n s_{n-1}$
- ($n-2$): $C_0 + C_1 s_1 + C_2 s_2 + \ldots + C_{n-2} s_{n-2} + C_{n-1} s_{n-2} + C_n s_{n-1}$
- $\vdots$
- (2): $C_0 + C_1 s_1 + C_2 s_2 + C_3 s_2 + \ldots + C_n s_{n-1}$
- (1): $C_0 + C_1 s_1 + C_2 s_1 + C_3 s_2 + \ldots + C_n s_{n-1}$
- (0): $C_0 + C_1 U + C_2 s_1 + C_3 s_2 + \ldots + C_n s_{n-1}$

The above expressions are numbered from 0 to n. Let $I(i)$ be the set of points $(s_1, s_2, \ldots, s_{n-1})$ ($U \geq s_1 \geq s_2 \geq \ldots \geq s_{n-1} \geq L$) that make expression ($i$) equal to 0. There must exist at least one i such that the measure of $I(i)$ is positive. According to Lemma 4, expression (i) must be constant 0.

If expression (0) is constant 0, then the total payment under mechanism t is 0 whenever the highest type is equal to the upper bound U. That is, for any θ, the total payment $C_0 + C_1[\theta]_1 +$

$C_2[\theta]_2 + \ldots + C_n[\theta]_n$ must be a constant multiple of $U - [\theta]_1$ (the total payment is a linear function). We have $C_0 = -UC_1$ and $C_j = 0$ for $j \geq 2$. It turns out that the above equalities of the C_j completely determine the values of the a_j, and the corresponding mechanism is the OEL mechanism with index $k = 0$ (details omitted).

Similarly, if expression (n) is constant 0, then the total payment under mechanism t is 0 whenever the lowest type is equal to the lower bound L (corresponding to the OEL mechanism with index $k = n$). If expression (i) is constant 0 for other i, then the total payment under mechanism t is 0 whenever the ith and $(i+1)$th type equal (corresponding to the OEL mechanism with index $k = i$). This finishes the proof. □

D Proof of Lemma 3

Note that in the public project problem for all $i \in \{1, \ldots, n\}$

$$g_i(\theta) = \sum_{j \neq i} v_j(f(\theta), \theta_j) = f(\theta)(\sum_{j \neq i} \theta_i - \frac{n-1}{n} c).$$

Let $h : [0, c]^{n-1} \to \mathbb{R}$ be the VCG mechanism (which is anonymous) and suppose that an anonymous feasible Groves mechanism $h' : [0, c]^{n-1} \to \mathbb{R}$ exists that welfare dominates h. So for all $\theta \in [0, c]^n$, $\sum_{i=1}^n h(\theta_{-i}) \leq \sum_{i=1}^n h'(\theta_{-i})$. We will show that then for all $x \in [0, c]^{n-1}$, $h'(x) = h(x)$.

Given $x \in [0, c]^{n-1}$, define $C(x) = |\{i : x_i = c\}|$, i.e., given a vector x of $n - 1$ types, $C(x)$ is the number of players who submitted c. Define the following predicate:

$$P(k): \ \forall x \in [0, c]^{n-1}(C(x) = k \to h'(x) = h(x)).$$

We now prove that $P(k)$ holds for all $k \in \{0, ..., n-1\}$, using induction (going backwards from $n - 1$). Let t and t' be the tax functions of the mechanisms h and h', respectively.

Base case.

Let x be such that $C(x) = n-1$. Consider $\theta := (c, ..., c) \in [0, c]^n$. Then for all $i \in \{1, \ldots, n\}$, $\theta_{-i} = x$. Clearly $f(\theta) = 1$ and no player is paying tax under the VCG mechanism. Hence for all $i \in \{1, \ldots, n\}$, $t_i(\theta) = g_i(\theta) + h(x) = 0$, where by the form of θ, $g_i(\theta) = g(\theta) = (n-1)(c - \frac{c}{n})$, so $h(x) = -g(\theta)$.

Since h' is a feasible mechanism and welfare dominates h,

$$0 = \sum_{i=1}^n t_i(\theta) \leq \sum_{i=1}^n t'_i(\theta) \leq 0,$$

so $\sum_{i=1}^n t'_i(\theta) = 0$. But

$$\sum_{i=1}^n t'_i(\theta) = \sum_{i=1}^n (g_i(\theta) + h'(x)) = n(g(\theta) + h'(x)).$$

Therefore $h'(x) = -g(\theta) = h(x)$.

Induction step.

Suppose $P(k)$ holds for some $k \geq 1$. We will prove $P(k-1)$. Let x be such that $C(x) = k-1$. The functions h and h' are permutation independent, so we can assume without loss of generality that the elements of x are sorted in descending order. Consider the type vector $\theta = (c, x)$, that is the concatenation of (c) and x. So θ starts with k c's and the rest is like the rest of x. Note that for $i \in \{1, \ldots, k\}$, $\theta_{-i} = x$ and $C(\theta_{-i}) = k-1$. For $i \in \{k+1, \ldots, n\}$, $C(\theta_{-i}) = k$ and by induction hypothesis, $h'(\theta_{-i}) = h(\theta_{-i})$ and hence $t'_i(\theta) = t_i(\theta)$.

Furthermore, $f(\theta) = 1$. We distinguish two cases.

Case 1. $k \geq 2$ or ($k = 1$ and $\sum_{j \neq 1} \theta_j \geq \frac{n-1}{n} c$).

In this case no player is paying tax under the VCG mechanism. Indeed, either there is another player who submitted c or if $k = 1$, the player does not alter the outcome. Hence, for all $i \in \{1, \ldots, n\}$, $t_i(\theta) = 0$, and because h' is feasible and welfare dominates h we get that $\sum_{i=1}^n t'_i(\theta) = 0$. We now look at $t'_i(\theta)$.

For $i \in \{1, \ldots, k\}$, $t'_i(\theta) = h'(x) + g_i(\theta)$, where $g_i(\theta) = g(\theta) = (k-1)(c - \frac{c}{n}) + \sum_{j=k+1}^n (\theta_j - \frac{c}{n})$. For $i \in \{k+1, \ldots, n\}$, by the induction hypothesis as mentioned above, $t'_i(\theta) = t_i(\theta) = 0$. Hence

$$\sum_{i=1}^n t'_i(\theta) = k(h'(x) + g(\theta)) = 0.$$

Therefore $h'(x) = -g(\theta)$ and it can be verified that this is precisely $h(x)$.

Case 2. $k = 1$ and $\sum_{j \neq 1} \theta_j < \frac{n-1}{n} c$.

Now the taxes under the VCG mechanism are different. For $i \in \{2, \ldots, n\}$, $t_i(\theta) = 0$ since the first player submits c. For $i = 1$, $t_1(\theta) = h(x) + g_1(\theta)$. Note that $h(x) = 0$. Hence $\sum_{i=1}^n t_i(\theta) = g_1(\theta)$. Now for $i \in \{2, \ldots, n\}$, $C(\theta_{-i}) = k$ and by the induction hypothesis, $t'_i(\theta) = t_i(\theta) = 0$. Hence $\sum_{i=1}^n t'_i(\theta) = t'_1(\theta)$. Since h' is feasible and welfare dominates h we have:

$$g_1(\theta) = \sum_{i=1}^n t_i(\theta) \leq \sum_{i=1}^n t'_i(\theta) = t'_1(\theta) = h'(x) + g_1(\theta) \leq 0.$$

Since $g_1(\theta) \geq 0$, this implies that $h'(x) = 0$ and hence $h'(x) = h(x)$. This completes the proof of the induction step and hence the proof of the lemma.

E Proof of Theorem 4

Suppose otherwise. Let h be the VCG mechanism for a public project problem, for some $c > 0$, some c_i, and $n \geq 2$, and let h' be a pay-only Groves mechanism that dominates it. Then, for some $\theta \in \Theta$ and $i_0 \in \{1, \ldots n\}$

$$g_{i_0}(\theta) + h_{i_0}(\theta_{-i_0}) < g_{i_0}(\theta) + h'_{i_0}(\theta_{-i_0}) \leq 0, \tag{3}$$

where the second inequality holds since h' is pay-only.

By definition of the VCG mechanism, for some $d_0 \in \{0, 1\}$ we have $h_{i_0}(\theta_{-i_0}) = -\sum_{j \neq i_0} v_j(d_0, \theta_j)$. Let θ' be θ with θ_{i_0} replaced by c_{i_0}. We now prove, by distinguishing two cases, that:

$$g_{i_0}(\theta') + h_{i_0}(\theta'_{-i_0}) = 0 \tag{4}$$

Case 1 $d_0 = 1$.

Then, by definition of h_{i_0}, $\sum_{j \neq i_0} v_j(1, \theta_j) \geq \sum_{j \neq i_0} v_j(0, \theta_j)$, i.e., by definition of v_j, $\sum_{j \neq i_0} \theta_j \geq c - c_{i_0}$. So $\sum_{i=1}^{n} \theta'_i \geq c$ and hence $f(\theta') = 1$. Note now that $g_{i_0}(\theta') = \sum_{j \neq i_0} v_j(1, \theta_j)$. Consequently $g_{i_0}(\theta') + h_{i_0}(\theta'_{-i_0}) = 0$ since $h_{i_0}(\theta'_{-i_0}) = h_{i_0}(\theta_{-i_0}) = -\sum_{j \neq i_0} v_j(1, \theta_j)$.
Case 2 $d_0 = 0$.

Then, by definition of v_j, $\sum_{j \neq i_0} \theta_j < c - c_{i_0}$. So $f(\theta') = 0$ and hence $g_{i_0}(\theta') = \sum_{j \neq i_0} v_j(0, \theta_j)$. Consequently $g_{i_0}(\theta') + h_{i_0}(\theta'_{-i_0}) = 0$ as well, since now $h_{i_0}(\theta'_{-i_0}) = h_{i_0}(\theta_{-i_0}) = -\sum_{j \neq i_0} v_j(0, \theta_j)$.

But h' dominates h and h' is pay-only, so

$$0 = g_{i_0}(\theta') + h_{i_0}(\theta'_{-i_0}) \leq g_{i_0}(\theta') + h'_{i_0}(\theta'_{-i_0}) \leq 0,$$

where the equality holds by (4). Hence $g_{i_0}(\theta') + h'_{i_0}(\theta'_{-i_0}) = 0$ and consequently $h_{i_0}(\theta'_{-i_0}) = h'_{i_0}(\theta'_{-i_0})$, i.e., by definition of θ', $h_{i_0}(\theta_{-i_0}) = h'_{i_0}(\theta_{-i_0})$. Thus we have a contradiction with (3).

F Proof of Theorem 5

We will show this for $n = 3$. For $n > 3$, it is fairly simple to extend the proof and we omit it from this version. The VCG mechanism is feasible, hence it suffices to show by Note 1(*ii*) that the VCG and BC mechanisms do not coincide, for some choice of c, c_1, c_2, c_3, with $c_1 + c_2 + c_3 = c$.

To this end we need to find θ_2 and θ_3 so that $S_1^{BCGC}(\theta_2, \theta_3) < 0$. Here

$$S_1^{BCGC}(\theta_2, \theta_3) := \max_{\theta'_1 \in \Theta_1} (L - (R_1 + R_2 + R_3)),$$

where for $\theta' := (\theta'_1, \theta_2, \theta_3)$

$$L := (n-1) \sum_{k=1}^{n} v_k(f(\theta'), \theta'_k),$$

$$R_1 = \max_{d \in D} \sum_{j \neq 1} v_j(d, \theta'_j) = \max\{0, \theta_2 + \theta_3 - (c_2 + c_3)\},$$

$$R_2 = \max_{d \in D} \sum_{j \neq 2} v_j(d, \theta'_j) = \max\{0, \theta'_1 + \theta_3 - (c_1 + c_3)\},$$

$$R_3 = \max_{d \in D} \sum_{j \neq 3} v_j(d, \theta'_j) = \max\{0, \theta'_1 + \theta_2 - (c_1 + c_2)\}.$$

Now, take $c = 100, c_1 = 10, c_2 = 40, c_3 = 50$ and $\theta_2 := 10$, $\theta_3 := 70$. Then $R_1 + R_2 + R_3 = \theta'_1 + 10 + \max\{0, \theta'_1 - 40\}$. Two cases arise.
Case 1 $f(\theta') = 0$.

Then $L = 0$, so $L - (R_1 + R_2 + R_3) \leq -10$.
Case 2 $f(\theta') = 1$.

Then $L = 2(\theta'_1 + \theta_2 + \theta_3 - 100) = 2\theta'_1 - 40$, so

$$\begin{aligned} L - (R_1 + R_2 + R_3) &= \theta'_1 - 50 - \max\{0, \theta'_1 - 40\} \\ &\leq (\theta'_1 - 50) - (\theta'_1 - 40) \leq -10. \end{aligned}$$

This proves that $S_1^{BCGC}(\theta_2, \theta_3) \leq -10$. By taking any $\theta'_1 \in [40, 100]$ we see that in fact $S_1^{BCGC}(\theta_2, \theta_3) = -10$.

By virtue of Theorem 4, the BC mechanism in the proof is not pay-only. Indeed, for the VCG mechanism there we have $t_1(0, \theta_2, \theta_3) = 0$, so that $t_1^{BCGC}(0, \theta_2, \theta_3) = 10/3$.