Ad Hoc Network Capacity

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Abstract

We study the performance of wireless ad hoc networks in the limit of a large number of nodes. We first define ad hoc networks and their capacity. We then calculate a lower bound on the capacity by constructing a communications scheme that achieves it. We also determine an upper bound on the capacity by investigating the properties of any transmission schedule. Bounds indicate that for a large number of nodes, the total network capacity increases with a rate between $k_1 n^{\frac{2}{3}}$ and $k_2 n^{\frac{1}{2}}$, while the per-node rate goes to zero.

1. Introduction

Wireless networks consist of a number of nodes communicating over a wireless channel. Depending on their architecture, they can be roughly divided in two categories. In those following the cellular paradigm, all nodes communicate exclusively with a base station that is responsible for controlling all transmissions and forwarding packets to the intended users. In those following the ad hoc paradigm, all nodes have the same capabilities and responsibilities. Two nodes wishing to communicate can do so directly, or else by using nodes lying in between them to route their packets.

In the past, research on ad hoc networks has mainly focused on the routing [4] and medium access control (MAC) [1] problems, and on issues regarding the physical layer [3]. The problem of determining theoretical upper bounds on the performance of ad hoc networks was not attacked until recently [2]. The authors of [2] studied the capacity of ad hoc networks in the limit of a large number of nodes. Here we extend the results of [2] to a 3-dimensional topology, and incorporate the Shannon capacity into the link model.

The rest of the paper is organized as follows: In Section 2, we provide definitions related to the ad hoc network. In Section 3 we develop a lower bound on the performance of ad hoc networks. An upper bound is derived in Section 4. We present conclusions in Section 5.

2. Definitions

Let $\{A_1, \ldots, A_n\}$ be a collection of $n$ nodes. Every node is equipped with a transmitter and a receiver and wants to communicate with another node chosen randomly, uniformly, and independently among the rest. Therefore, there exist $n$ communicating pairs of nodes. We call the first element of the pair the source and the second the sink. The nodes are uniformly and independently distributed within the cube $C = \{ (x, y, z) : |x|, |y|, |z| \leq \frac{1}{2} \}$ of side 1 and volume 1. Each node, at a given time, may remain idle, receive data from a single node, transmit information to its sink, either directly or through another node, or relay information coming from other nodes.

Each node transmits with power $P$, and link rate $\lambda$, using the whole system bandwidth $W$. Furthermore, total synchronization is assumed, so that all transmitting nodes start sending out new bits at exactly the same time. The power received by a node $A_i$ that lies at a distance $d$ from a transmitting node $A_j$ will be modeled as $P_{ij} = K \frac{d}{d^2}$ where $K$ is constant for all receivers.

At any time $t$ each receiving node will receive a number of signals only one of which is intended for that node. It will attempt to demodulate this signal and treat the rest as interference. We assume that the other signal powers will add incoherently to produce a total interference power $I$. Furthermore each receiver is hampered by thermal noise with power $N_0$. Therefore, each receiving node $A_j$ is characterized by a Signal to Interference and Noise Ratio (SINR) given by $\gamma_j = \frac{P_{ij}}{N_0 + I}$. We assume the receiver can retrieve the signal if $\gamma_j$ exceeds or is equal to a threshold value $\gamma_T$ common for all receivers. If not, the signal is lost. The threshold value $\gamma_T$ is connected to the link rate $\lambda$ by the Shannon capacity formula $\lambda = W \log_2 (1 + \gamma_T)$. Although we will use this formula, the analysis can trivially be generalized in the case where the rate and the SINR are connected by a formula derived from inverting the $P_b$ expression of a specific modulation scheme. We formally define as an ad hoc network any collection of nodes for which the above assumptions hold.

We will say that a source-sink pair $(A_i, A_j)$ will com-
communicate with a source rate \( \Lambda_{ij} \) under a communications scheme, if, under this scheme, there is a \( T \) such that during the time interval \([0, T]\), \( \Lambda_{ij} T \) bits are transferred from \( A_i \) to \( A_j \). We call a source rate \( \Lambda \) uniformly achievable if there is a communications scheme that allows all nodes to communicate with their intended sinks using this rate. We define the capacity \( C(n) \) of a network of size \( n \) to be the supremum of the uniformly achievable rates for this network, multiplied by the number of nodes \( n \). Therefore the network capacity is an upper bound on the aggregate throughput of the network if all source-sink pairs use the same rate for their communication. Since the capacity of a network will depend on its topology and the topology is generally random, the capacity will be a random variable. As in [2], we present bounds on the ad hoc network capacity that will hold with high probability in the limit of a large number of nodes.

In the following we will often study sequences of events \( E(n) \) whose probability approaches unity as the number of nodes \( n \) goes to infinity. We say that such events occur in the limit of large \( n \).

3. Capacity lower bound

We derive a lower bound on the capacity of ad hoc networks by constructing a communication scheme that achieves that bound. Under this scheme, nodes will use routing and time division to forward packets to their sinks. The lower bound will be shown to hold in the limit of large \( n \).

3.1. Geometric partition into cells

We consider an ad hoc network with \( n \) nodes. We start by dividing the cube \( C \) into a lattice of \( L(n) \) cubic cells, which we index by \( c_1, c_2, \ldots, c_{L(n)} \). The cells cover the cube completely and do not overlap. Their boundaries are along the \( 3(m + 1) \) planes

\[
\begin{align*}
    x & = \frac{1}{2}, x = -\frac{1}{2} + \frac{1}{2m}, \ldots, x = \frac{1}{2} \\
    y & = \frac{1}{2}, y = -\frac{1}{2} + \frac{1}{2m}, \ldots, y = \frac{1}{2} \\
    z & = \frac{1}{2}, z = -\frac{1}{2} + \frac{1}{2m}, \ldots, z = \frac{1}{2}
\end{align*}
\]

with \( m = L(n)^\frac{1}{3} \). We require that the total number of cells is related to the number of nodes. Specifically, we require that there is a \( n_0 \) such that \( L(n) \) satisfies the inequality

\[
\frac{n}{4 \log(n)} < L(n) < \frac{n}{2 \log(n)} \quad \forall n > n_0 \tag{1}
\]

where \( \log(x) \) denotes the natural logarithm of \( x \). Although the total number of cells must be the third power of an integer for all \( n \), this requirement can be satisfied for \( n > n_0 \) if \( n_0 \) is large enough, given that \( \frac{n}{\log(n)} \to \infty \).

Equation (1) links the number of nodes with the number of cells and leads to the following lemma.

**Lemma 1** In the limit of large \( n \), all cells will contain at least one node.

**Proof:** Each node is placed randomly, and with uniform distribution in one of the cells. Nodes are also placed in the cells independently of each other, so the probability that cell \( c_1 \) will be empty once all nodes are placed will be

\[
Pr\{c_1 \text{ is empty}\} = (1 - \frac{1}{L(n)})^n \tag{2}
\]

We then have:

\[
\begin{align*}
Pr\{\exists j : c_j \text{ is empty}\} & \leq \sum_{j=1}^{L(n)} Pr\{c_j \text{ is empty}\} \\
& = nPr\{c_1 \text{ is empty}\} \\
& = n(1 - \frac{1}{L(n)})^n \tag{3}
\end{align*}
\]

The inequality comes from the union bound. The first equation comes from symmetry, and the second from Equation (2). However, we note that the natural logarithm of the right size goes to \(-\infty \) for large \( n \):

\[
\log\{n(1 - \frac{1}{L(n)})^n\} = \log(n) + n \log(1 - \frac{1}{L(n)}) \\
\leq \log(n) - \frac{n}{L(n)} \\
\leq \log(n) - 2 \log(n) \\
= -\log(n) \\
\to -\infty. \tag{4}
\]

and the result is proven. The first inequality of (4) comes from the fact that \( \log(1 + x) \leq x \). The second inequality comes from Equation (1).

Therefore, in the limit of large \( n \) all cells will contain at least 1 node. We select one of them randomly and call it the cluster head of the cell.

3.2. Routing strategy

We require that nodes only transmit to (and receive from) nodes that lie in adjacent cells. Therefore, if the sink of a node is more than one cell away, the node will have to route the information through the cluster heads of the cells that lie in between. We define the route of a source to be the
collection of all cells that the source will use to forward packets to its sink, including the cells of the source and the sink. Using Equation (1) we can establish the following lemma. Its proof is straightforward but lengthy, so we omit it.

**Lemma 2** Let \( l_j \) be the number of routes to which cell \( c_j \) belongs. In the limit of large \( n \),

\[
l_1, l_2, \ldots, l_L(n) \leq l_{\text{max}}(n) = 2000n^{\frac{1}{2}}(\log(n))^{\frac{1}{2}}
\]  

(5)

### 3.3. Time division strategy

Because cells have the shape of cubes, each cell will be adjacent to exactly 26 other cells. Therefore we can decompose the lattice of cells into 27 sets of cells, such that no cells lying in the same set have a common boundary. We divide the time into frames of 27 slots each. Each set of cells is assigned to a slot. Furthermore, at any slot only one node from each cell belonging to the set can transmit. This time division ensures that no receiver is overwhelmed by an interfering transmitter that may be too close, and leads to the following bound. The proof is lengthy, so we omit it.

**Lemma 3** The interference experienced by any receiver increases at most linearly with the number of cells. Specifically: \( I < 450KPL(n) \) for all topologies. In addition, there is an \( n_0 \) such that communication between neighboring nodes is feasible for all topologies if \( n > n_0 \) and the rate is set to

\[
\lambda(n) = \frac{W}{2000} \left( \frac{\log(n)}{n} \right)^{\frac{1}{2}}.
\]  

(6)

### 3.4. Final bound

We now combine the results of the previous subsections to arrive at the capacity lower bound:

**Theorem 1** There is a positive constant \( k_1 \) such that

\[
\lim_{n \to \infty} \Pr \{ C(n) \geq k_1 \frac{n^{\frac{1}{2}}}{\log(n)} \} = 1
\]  

(7)

**Proof:** We have shown that every cell will contain a cluster head (Lemma 1). We have also shown that in the limit of large \( n \) all cluster heads will need to accommodate at most \( l_{\text{max}}(n) = 2000(\log(n))^{\frac{1}{2}}n^{\frac{1}{2}} \) different routes (Lemma 2). Finally, we have shown that each cluster head can receive information with a rate at least equal to \( \lambda(n) = \frac{W}{2000} \left( \frac{\log(n)}{n} \right)^{\frac{1}{2}} \) (Lemma 3). However, because of the time sharing scheme, each receiver will be active for \( \frac{T}{27} \) seconds in every frame. Therefore, every path can be serviced with a source rate

\[
\Lambda(n) = \frac{\lambda(n)}{27l_{\text{max}}(n)} = \frac{k_1}{\log(n)n^{\frac{1}{2}}}, \quad k_1 = \frac{1}{27 \times 2000^{\frac{1}{2}}}
\]  

(8)

This rate can be achieved with probability approaching 1 as \( n \to \infty \). Since the capacity of the network is the supremum of all achievable rates, multiplied by \( n \), it will exceed the uniform rate of Equation (8) multiplied by \( n \). Thus \( C(n) \geq n\Lambda(n) \), which gives the desired result.

\[ \blacksquare \]

### 4. Capacity upper bound

In this section we derive an upper bound on the capacity of ad hoc networks. Our approach is to show that all schemes are subject to a set of fundamental constraints that limit their performance.

We consider an ad hoc network with \( n \) nodes indexed by \( A_i, i = 1, \ldots, n \). We assume that the network uses a communication scheme that allows each source to talk to its sink with a uniform source rate of \( \Lambda(n) \) bits/sec. The assumptions of Section 2 state that the nodes will use a common link rate equal to \( \lambda(n) \), that requires an SINR \( \gamma(n) \) for a successful transmission. The SINR and the link rate are connected by \( \lambda(n) = W \log_2(1 + \gamma(n)) \). During the time interval \([0, T]\), a total of \( H \) transmissions of single bits between two nodes takes place. Each transmission is over a distance \( d_h, h = 1, \ldots, H \).

### 4.1. Division of time into slots

As discussed in the definition of the source rate, during the time interval \([0, T]\) each node succeeds in sending \( TA(n) \) bits of information to its sink. Because of the synchronization that exists, we can also assume that \( \lambda(n)T \) is an integer (otherwise in the last \( \lambda(n)T - \lfloor \lambda(n)T \rfloor \) seconds no bits were transferred). Let \( M = \lfloor \lambda(n)T \rfloor \). We break the time interval \([0, T]\) into \( M \) slots \( s_l, l = 1, \ldots, M \). During each slot, a given and fixed set of nodes transmits completely a single bit. Let \( T_1 \) be the total amount of time during which only one transmission was active. Let \( T_2 = T - T_1 \). Obviously both \( T_1 \) and \( T_2 \) will be multiples of the slot duration. We also assume that all transmissions were successful. Otherwise, all unsuccessful transmissions could be canceled, and this would not reduce the uniform source rate and would also lower the power of the interference at the rest of the nodes.

### 4.2. Asymptotic properties of ad hoc networks

We define the transport distance \( X_i \) of a node \( A_i \) to be the distance between the node and its sink. We define the aggregate transport distance \( X(n) \) to be the sum of
all transport distances: \( X(n) = \sum_{i=1}^{n} X_i \). The transport distances are not independent random variables since, for example, two nodes may have the same sink. However, as the number of nodes increases the correlation between any two of them decreases “fast enough”. Therefore the following result will hold. The proof is omitted because of space limitations.

**Lemma 4** In the limit of large \( n \), \( X(n) \geq \frac{n}{12} \). Since the relaying nodes are not optimally placed, we will also have:

\[
\sum_{h=1}^{H} d_h > \frac{nA(n)T}{12} \tag{9}
\]

We denote with \( S(n) \) the minimum distance between any two nodes of an ad hoc network of size \( n \): \( sp(N) = \min_{A_i, A_j, i \neq j} d_{st}(A_i, A_j) \). \( S(n) \) is a positive random variable whose distribution concentrates around zero as the number of nodes increases. However, as the following lemma shows, this happens with finite rate.

**Lemma 5** In the limit of large \( n \), \( S(n) \geq a(n) \) with

\[
a(n) = \frac{1}{n^{\frac{3}{2}}(\log n)^{\frac{3}{4}}} \tag{10}
\]

Since the received power depends on the distance, this inequality can be translated into an upper bound on the receiver SINR \( \gamma_j \) of any receiving node \( A_j \):

\[
\gamma_j(n) \leq \frac{KP}{N_0} n^{\frac{3}{2}}(\log(n))^{\frac{3}{4}} \tag{11}
\]

**Proof:** Consider a region around every node that is the intersection of the cube \( C \) with a closed sphere where the sphere is centered on the node and has a radius of \( a(n) \). The volume of the intersection will be less than \( b(n) = \frac{4\pi}{3} a^3(n) \). \( S(n) \) will be greater than \( a(n) \) if no node lies within any of these intersections (other than the one on their center). Therefore:

\[
\Pr\{sp(N(n)) > a(n)\} \geq (1 - b(n))... (1 - (n - 1)b(n)) > (1 - nb(n))^n \tag{12}
\]

We can also trivially show that there exists a \( x_0 \) such that \( \log(1 + x) \geq 1.1x \) for \( x_0 < x < 0 \). Since \( nb(n) \) will converge to 0 from negative values, we have:

\[
\log(1 - nb(n))^n = n \log(1 - nb(n)) \geq - \frac{1.1}{4\pi} \frac{4\pi}{\log n} \frac{3}{4} \tag{13}
\]

which tends to zero for large \( n \). Combining the above:

\[
\Pr\{sp(N(n)) > a(n)\} \to 1, \text{ for } n \to \infty \tag{14}
\]

Equation (11) can then be trivially derived, since the upper bound is the SINR when the receiver lies \( a(n) \) meters away from the transmitter and there is no interference.

The following lemma represents a bound on how large all the transmission lengths can be, given that they cause interference to the rest of the transmissions:

**Lemma 6**

\[
\sum_{h=1}^{H} d_h^2 \leq 3\lambda(n)T_1 + \frac{6T_2\lambda(n)}{\gamma(n)} \tag{15}
\]

**Proof:** During slot \( l \), \( Q(l) \) transmissions take place. The transmission distances are \( d_{jq} \), with \( q = 1, \ldots Q(l) \). Each transmission is hampered by \( Q(l) - 1 \) interfering transmissions. Let \( d_{jq} \) be the distances of the interfering transmitters from the receiver, with \( r = 1, \ldots Q(l) - 1 \).

For a given slot and transmission, the following will hold, since we assumed all transmissions are successful.

\[
P \frac{KP}{d_{jq}^2} \{ N + \sum_{r=1}^{Q(l)-1} \frac{KP}{(d_{jq})^2} \}^{-1} \geq \gamma(n) \Rightarrow
\]

\[
\frac{P}{d_{jq}^2} \geq \gamma(n) \left\{ \frac{N}{K} + \sum_{r=1}^{Q(l)-1} \frac{P}{(d_{jq})^2} \right\} \Rightarrow
\]

\[
d_{jq}^2 \leq \frac{P}{\gamma(n)} \left[ \frac{1}{K} + \sum_{r=1}^{Q(l)-1} \frac{P}{(d_{jq})^2} \right] \Rightarrow
\]

\[
d_{jq}^2 \leq \frac{P}{\gamma(n)} \left[ \frac{1}{K} + \frac{Q(l)-1}{n} \right] \tag{16}
\]

We can then sum over all transmission distances in the slot, and derive

\[
\sum_{q=1}^{Q(l)} d_{jq}^2 \leq Q(l) \frac{P}{\gamma(n)} \frac{1}{K} + \left( Q(l) - 1 \right) \frac{1}{n} \tag{17}
\]

In the case \( Q(l) > 1 \) we can then simply write

\[
\sum_{q=1}^{Q(l)} d_{jq}^2 \leq \frac{6}{\gamma(n)} \tag{18}
\]

In the case \( Q(l) = 1 \) (only one active transmitter) another identity is readily available:

\[
\sum_{q=1}^{Q(l)} d_{jq}^2 \leq 3 \tag{19}
\]

For the \( T_2\lambda(n) \) slots during which more than one transmission is taking place, Equation (18) holds. For the rest, Equation (19) will hold. Combining the inequalities and summing over all slots, we arrive at (15).
4.3. Final bound

We now combine the results of the previous subsections to arrive at an upper bound for the capacity, in the limit of large \( n \).

**Theorem 2** There is a positive \( k_2 \) such that

\[
\lim_{n \to \infty} \Pr \{ C(n) \leq k_2 \log(n) n^{\frac{1}{2}} \} = 1 \tag{20}
\]

**Proof:** We first use the convexity of the \( f(x) = x^2 \) function to derive

\[
\left( \sum_{h=1}^{H} d_h^2 \right)^{\frac{1}{2}} \leq \sum_{h=1}^{H} \frac{1}{H} d_h^2 \tag{21}
\]

Also, at any given time not more than \( \frac{n}{2} \) users can be active. So

\[
H \leq \lambda(n) T n^{\frac{1}{2}} \tag{23}
\]

We then have

\[
\frac{n}{2} \Lambda(n) T
\]

\[
\leq \sum_{h=1}^{H} d_h
\]

\[
\leq H^{\frac{1}{2}} \left\{ \sum_{h=1}^{H} d_h^2 \right\}^{\frac{1}{2}} \tag{22}
\]

\[
\leq \left\{ \lambda(n) T n^{\frac{1}{2}} \right\}^{\frac{1}{2}} \left\{ \sum_{h=1}^{H} d_h^2 \right\}^{\frac{1}{2}} \tag{21}
\]

\[
\leq \left\{ \lambda(n) T n^{\frac{1}{2}} \right\}^{\frac{1}{2}} \left\{ 3 \lambda(n) T_1 + \frac{6 T_2 \lambda(n)}{\gamma(n)} \right\}^{\frac{1}{2}}
\]

\[
\leq \left\{ \lambda(n) T n^{\frac{1}{2}} \right\}^{\frac{1}{2}} \left\{ 4 \lambda(n) T_1 \right\}^{\frac{1}{2}}
\]

\[
= W T \log_2 (1 + \gamma(n)) \sqrt{2n}
\]

\[
\leq W T \log_2 n + \sqrt{2n}
\]

\[
= \frac{5 \sqrt{2}}{3 \log(2)} n^{\frac{1}{2}} \log n \tag{24}
\]

The first inequality comes from Lemma 4 and is only true in the limit. The second comes from (22). The third from (23) and the fourth from Lemma 6. The fifth inequality is derived from the fact that the function \( \frac{\Lambda(n)}{n} \) is bounded (trivial to prove) and the last inequality follows from Lemma 5.

We finally arrive at

\[
\Lambda(n) \leq k_2 \frac{1}{\sqrt{n}} \log n, \quad k_2 = \frac{20 \sqrt{2}}{\log(2)} W \tag{25}
\]

The above bound applies for all uniformly attainable source rates. Multiplying by \( n \), we derive the bound on capacity (20).

5. Discussion

We studied the capacity of ad hoc networks in the limit of a large number of nodes under a 3-dimensional topology with link rates governed by the Shannon capacity. We found that the capacity \( C(n) \) follows the inequality

\[
k_1 \frac{n^{\frac{1}{3}}}{\log(n)} \leq C(n) \leq k_2 \log(n) n^{\frac{1}{2}} \tag{26}
\]

with probability approaching unity as \( n \to \infty \), and \( k_1, k_2 \) some positive constants. The lower bound was determined by constructing a communications scheme that achieves it. The upper bound was derived by using inequalities that are common to all communication schemes.

Equation (26) suggests that, although the capacity increases with the number of users, the per user available rate decreases. Therefore, networks in which a minimum per user rate is required cannot be arbitrarily large. Also, the use of spatial separation in our constructive scheme guarantees that the per user available rate does not decrease as fast as if only time division was used. Indeed, if only time division is used, a single user will be active at any time, transmitting directly to its intended receiver, and its source rate will be inversely proportional to the number of nodes).

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References


