Optical Routing in Massively Dense Networks: Practical Issues and Dynamic Programming Interpretation

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Abstract—A form of routing based on an analogy with Geometrical Optics has recently been proposed for use in wireless networks with a very large number of nodes. The analogy hinges on taking a macroscopic view of the network, and describing it solely in terms of a cost function, that is analogous to the refractive index of Optics. In this setting, optimal routes resemble rays of light propagating through a properly defined optical medium. This method has intuitive appeal and leads to insightful results; nevertheless many issues remain unresolved. Two of these issues are treated here. First, we discuss the problem of route discovery, also considering the challenging cases where the optimal routes starting from a given node intersect themselves, and also fail to cover the whole network in a reasonable manner. Then, we show how optical routing can be derived using Dynamic Programming. The derivation illuminates the connection between this form of routing and traditional routing algorithms that are also based on Dynamic Programming.

Keywords: Dynamic Programming, Massively Dense Networks, Optics, Routing, Wireless Networks

I. INTRODUCTION

This work focuses on the problem of routing in wireless networks with a very large number of nodes. In this regime, most traditional routing protocols suffer from the scalability problem: as the number of nodes in the network increases, the routes become longer, and so they are more susceptible to node mobility and channel fading, and are harder to discover and maintain.

An imaginative solution to the scalability problem was originally proposed in [1], and later refined and extended in [2], [3]. There, it is assumed that the network has so many nodes, that, in addition to the usual microscopic view, a macroscopic view of the network also starts to emerge. It is shown that, under the macroscopic view, optimal routes connecting two points resemble rays of light connecting the same points in a properly defined optical medium.

In [1], [2], [3], the analogy between networking and optics was explored to a significant extend. Nevertheless, two important issues were left unexplored. Firstly, the theory of [1], [2], [3] gives a characterization of the optimal route connecting two points. However, the practical problem of how the nodes will discover this route in a distributed manner is not addressed. Note that, in traditional settings, this problem is well understood and investigated.

Secondly, and on the theoretical front, we establish the optimality of optical routing by means of Dynamic Programming. The appeal of this approach is that it shows more clearly the connection between optical routing and typical microscopic routing protocols, that are typically established also by Dynamic Programming techniques.

The rest of this work is organized as follows: In Section II we summarize the results of [1], [2], [3]. In Section III we discuss practical issues related to route discovery. In Section IV we discuss connections to Dynamic Programming. We conclude in Section V.

II. OVERVIEW OF OPTICS ANALOGY

In this section, we briefly review the results of [1], [2], [3], and in particular those aspects that are related to our work.

The basic idea is that, as the number of nodes in the network increases, we can take a macroscopic, i.e. bird’s eye view of the network, in addition to the standard microscopic view. The macroscopic view is not as detailed as the microscopic view, but nevertheless captures fundamental aspects of the network, and so permits a meaningful, if not exceedingly detailed, view of the network. In particular, we are no longer interested in the particular positions of nodes, since there are so many of them. Instead, in any location \( r \) of coordinates \((x, y)\) we describe the network in terms of the node density \( \lambda(r) \), which is measured in nodes per square meter. In addition, we are no longer interested in the cost of communication between particular nodes, but rather the cost function, defined as

\[
\epsilon(r) \triangleq \lim_{\epsilon \to 0} \frac{dc(r)}{\epsilon}
\]

where \( dc(r) \) is the incremental cost of transmitting the packet over an incremental distance \( \epsilon \), at location \( r \). Therefore, the cost of transmitting a packet over a non-incremental distance,
along a route \( R \), from point \( A \) to point \( B \), is the line integral

\[
[AB]_R = \int_A^B c(\mathbf{r}) \, d\mathbf{r}
\]

(1)
taken along the route \( R \).

A very natural question to ask is: what is the optimal route \( R_{\text{opt}} \), that connects two points \( A \) and \( B \) while incurring the minimum cost? In this setting, it can be shown that this optimal route is described by the same equation that describes the propagation of a ray of light connecting the two points, if the network is removed, and substituted by an optical medium of refractive index \( n(\mathbf{r}) = c(\mathbf{r}) \). This equation is:

\[
\frac{d}{ds}(n \, d\mathbf{r}) = \nabla n,
\]

(2)

where \( \mathbf{r} = (x, y) \) is the position vector of a typical point on the ray, \( s \) is the length of the ray measured from a fixed point on it, \( \mathbf{ds} \) is the direction on the ray at \( \mathbf{r} \), \( n(\mathbf{r}) \) is the refractive index at \( \mathbf{r} \), and \( \nabla n \triangleq \frac{\partial n}{\partial x} + \frac{\partial n}{\partial y} \) is its gradient.

A sketch of the proof that optimal routes also satisfy (2) with \( n(\mathbf{r}) = c(\mathbf{r}) \) is as follows: the aggregate cost over a route is given by (1). In that equation, if \( c(\mathbf{r}) \) is taken to be the refractive index of an optical medium, the cost becomes the optical length of a curve \( R \). A ray of light always satisfies (2) and, by the principle of Fermat [9], always has a locally minimum optical length, with respect to all other curves that are created by local perturbations of \( R \). An optimal route must have the minimum cost with respect to all its perturbations, and so must also satisfy (2).

In general, there can be multiple rays \( R_1, \ldots, R_n \) connecting two points (anyone wearing glasses or using magnifying lenses knows this). All of them satisfy (2), but with different initial conditions. Each of these rays has an optical length \( \int_A^B n(\mathbf{r}) \, d\mathbf{r} \) which is a local minimum with respect to perturbations of that ray. Clearly, one of these rays will correspond to the global minimum.

Until now, we have not specified the functional form of the cost function. Different choices are possible, depending on what we want to optimize in the network:

1. In [1], the author implicitly uses the cost function \( c(\mathbf{r}) = \sqrt{\lambda(\mathbf{r})} \), where \( \lambda(\mathbf{r}) \) is the node density. This form corresponds to a network in which we want to send data from point \( A \) to \( B \) using the minimum number of hops, and communication is constrained between nearest neighbors.

2. In [3], we propose the cost function \( c(\mathbf{r}) = \frac{1}{\sqrt{\lambda(\mathbf{r})}} \). This form corresponds to a network with limited bandwidth, so that the aim is to send information from \( A \) to \( B \) while consuming the minimum amount of bandwidth.

3. In [3], we also propose the cost function \( c(\mathbf{r}) = f(\lambda(\mathbf{r})) \), where \( f(x) \to \infty \) for \( x \to 0 \), and \( f(x) \to \text{const} \) for large values of \( x \). This is shown to be a good choice where the aim is to minimize the energy needed for transporting packets from \( A \) to \( B \), and the energy cost of a single transmission over some distance \( d \) is \( ad^b + c \), where \( a, b, c \) are constants.

4. If the cost function is taken to be constant, i.e., \( c(\mathbf{r}) = \text{const} \), then optimal routes resemble rays of light in a homogeneous medium where the refractive index is constant, i.e., the optimal routes become straight lines. This case corresponds to a setting where routing is equally costly at all parts of the network, so we would like to minimize the length of routes.

As is intuitively clear, the precise choice of cost function can dramatically affect the shape of optimal routes. For example, in Fig. 1 we draw the optimal routes for the same node density \( \lambda(x, y) = (3 \times 10^{-5}x^2 + 0.01) \text{ nodes/m}^2 \), but for the four different cost functions described above: the cost function of [1] (Route R1), the constant cost function (route R2), the cost function that leads to energy efficient communication (route R3), and the cost function that corresponds to bandwidth efficiency (route R4). In the figure, areas of higher node density are denoted with darker shades of gray.

III. ROUTE DISCOVERY

Work in [1], [2], [3] has focused on the theoretical characterization of the optimal route connecting two points \( A \) and \( B \). The important practical problem of how the route is determined by the nodes \( A \) and \( B \) in a distributed fashion has not been addressed. In this section, we address this problem.

Let us first consider the following protocol for route establishment (we note that similar protocols have appeared in the context of position based routing [10]): As shown in Fig. 2, any nodes \( A, B \) interested in exchanging packets, will launch a number \( N \) of route request (RRQ) packets that will propagate along rays starting at them, and with equally spaced initial angles.

A number of comments are in order: Clearly, the larger the number \( N \) of launched rays, the denser will be the covering of the network. On the other hand, note that, due to the inhomogeneity of the medium, the rays will not cover the
network uniformly. Finally, note that the ray propagation can be performed locally. When the packets reaches a location of the network $r$, the nodes in that location will impose an incremental change in the direction of the ray, according to (2), and then will propagate the ray further. Therefore, and in contrast to similar schemes, such as Trajectory Based Forwarding [11], there is no need for the node $A$ to determine completely the whole route, by locally solving the differential equation (2).

If nodes $A$ and $B$ propagate RRQ packets in this manner, the emanating rays will inevitably intersect with each other. Nodes that lie at the intersection points will realize that nodes $A$ and $B$ want to communicate, and will use the established routes to inform each of the nodes about the location of the other party.

Nodes $A$ and $B$ now have two options:

1) After receiving notification from all locations where the emanating rays intersect, they will have an idea about the physical location of the other nodes, and so will launch additional rays, with the aim of finding the initial angle that can reach the other party.

2) $A$ and $B$ may choose not to launch additional rays, but rather chose to transport their packets using different segments of the already established rays. In this case, packets will go through one of the locations where the rays intersect. As a result, packets will not use an optimal route, but rather two segments of optimal routes. The resulting route may be sufficiently good, and avoids the need to launch more rays. As in this case routing is always along rays, which represent geodesics [12] for the system, we will call this variant of our protocol geodesic routing.

The discussion until now has implicitly assumed that rays emanating from nodes do not intersect with each other. However, this is typically not the case. As an example, in Fig. 3 we have plotted 20 rays emanating from the location $(2,0)$, in a medium where the refractive index as a function of the position is given by the equation $n(x, y) = \frac{1}{\sqrt{x^2+y^2}+0.01}$. Rays initially diverge from each other in a uniform manner, but due to the fact that rays tend to bend toward areas with high values for the refractive index, after they travel some distance they start to intersect.

As a result of the rays intersecting, it is clear that the routing protocol we have developed until now requires modification. It continues to be true that, for any two points $A$ and $B$, there is a ray connecting them that incurs the minimum cost of communication. If, however, we are restricted to use a given set of rays emanating from the points $A$ and $B$, then it is possible that the optimal route will consist of multiple segments of rays. For example, the optimal route between the point $(2,0)$ and the point $A$ noted in the figure consists of no less than four ray segments, which are shown in the figure. Therefore, in order to find the minimum cost route we will need to consider all possible combinations of ray segments.

Another complication arises from the fact that rays emanating from a point uniformly, will not cover the network uniformly. One such example appears in Fig 4. In the figure, 20 rays emanate uniformly from the origin, and the refractive index as a function of the location $(x, y)$ is given by $n(x, y) = 2\exp(-y^2)$. In the figure, it is clear that the rays tend to concentrate on the strip where the refractive index has large values, and only six rays manage to escape the pull of the strip. Clearly, the network is not covered uniformly. A potential solution is to expand the set of original emanating rays by an additional set of rays, that emanate from positions along those of the original emanating rays that managed to penetrate areas of the network with low refractive index. This is easy to perform in a distributed manner: When a ray enters areas where the refractive index (equivalently cost) is much smaller than from where it started, it is in a part of the network that is not covered well, and so additional rays must be launched. An example appears in Fig. 5.
IV. DYNAMIC PROGRAMMING

A. Background on Dynamic Programming

Dynamic Programming (DP) deals with problems where decisions are made in stages [13]. The aim is to minimize a certain cumulative cost function, which models the accumulation of cost during the time evolution of the system. As the system evolves, decisions are made on the basis of the present state and the expected cost for the next stage.

Without loss of generality, we represent the evolution of the system in the form:

\[ x_{k+1} = f_k(x_k, u_k, w_k); \quad k = 0, 1, \cdots, N - 1 \]

where

- \( k \in \mathbb{N} \) indexes discrete time.
- \( x_k \in S_k \) is the state of the system and summarizes past information that is relevant for future optimization.
- \( u_k \) is the control or decision variable at time \( k \).
- \( w_k \) is a random disturbance (or noise).
- \( N \in \mathbb{N} \) is called the horizon and is the number of times we apply the control function, i.e., we make a decision.
- \( f_k \) describes the system and in particular the mechanism to update the state.

The aforementioned cost function is additive over time, i.e., the cost incurred at time \( k \), \( g_k(x_k, u_k, w_k) \), is summed up with all the previous costs; hence the function that must be minimized is:

\[ J(x_0) = E\left\{ g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k, w_k) \right\} \]  

(4)

where \( g_N(x_N) \) is a terminal cost incurred at the last stage of the process and \( x_0 \) is the starting point of the process. The main goal of DP is the minimization of (4). This formulation is very general and can be applied to a wide range of problems across various disciplines such as mathematics, economics, etc. Here, we are interested in its application in (i) routing and the minimization of routing cost and (ii) Geometrical Optics and the minimization of optical length.

B. Dynamic Programming and Shortest Path Problems

Consider a network of \( N \) nodes in which a certain node \( D \) is the destination node and we are interested in finding the shortest path from node \( x^* \) to node \( D \). Obviously the shortest path will be composed of no more than \( N - 1 \) hops.

We denote as \( d_k(x^*) \) the communication cost required to transfer a packet from node \( x^* \) to \( D \) in \( (N - k) \) hops. It can be shown, by standard DP arguments, [13], that \( d_k(x^*) \) can be generated by the iteration:

\[ d_k(x^*) = \min_{x \in V(x^*)} [d_{k+1}(x) + d(x, x^*)] \]  

(5)

where \( V(x^*) \) is the set of neighbors of \( x^* \). This is called Bellman’s equation and means that in order to find the minimum hop path, we must take the minimum, over all neighbors \( x \), of the sums of the length \( d(x, x^*) \) of the arc between node \( x \) and \( x^* \), plus the shortest path length \( d_{k+1}(x) \) to node \( D \).

C. Dynamic Programming and Geometrical Optics

Dynamic Programming can be used to derive the fundamental equation of geometrical optics, known as the eikonal equation, which in term gives the equation (2) describing the propagation of light rays.

To see this, let a source of light be placed at some point \((x_0, y_0)\), and let

\[ S(x, y) = \min_{\theta} \int_{C} n(x(s), y(s)) \, ds \]  

(6)

be the minimum optical length over all curves starting from \((x_0, y_0)\) and ending at \((x, y)\). As discussed, the principle of Fermat specifies that any ray of light that starts at \((x_0, y_0)\) and goes to \((x, y)\) achieves a local minimum.

This minimization can be performed using the traditional tools of calculus of variation but it can be shown, [14], that also DP can be applied. In particular, applying backward DP to this problem, [13], [14], we consider an infinitesimal path \( \Delta s \) between the points of coordinates \((x - \Delta s \sin \theta, y - \Delta s \cos \theta)\) and \((x, y)\) and along a locally optimal path \( C \). Hence, we can state the following minimization functional equation:

\[ S(x, y) = \min_{\theta} [S(x - \Delta s \sin \theta, y - \Delta s \cos \theta) + n(x - \Delta s \sin \theta, y - \Delta s \cos \theta) \Delta s + o(\Delta s)] \]  

(7)
As $\Delta s \to 0$, $S(x - \Delta s \sin \theta, y - \Delta s \cos \theta) \approx S(x, y) - \sin \theta \Delta s \frac{\partial S}{\partial x} - \cos \theta \Delta s \frac{\partial S}{\partial y}$. It is then straightforward to show that the minimization is achieved for the angle $\theta$ for which $\tan \theta = \frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y}^{-1}$. After some simple algebra, we arrive at:

$$S_x^2 + S_y^2 = n^2(x, y). \quad (8)$$

The above equation is known in Optics as the eikonal equation. It readily gives (2) after a few manipulations [9].

**D. Optimal routing and geometrical optics: formal analogy**

In Section IV-A we reviewed the basic concepts of DP which can be used to minimize the cost associated to the evolution of a discrete system in a wide range of scenarios. In Section IV-B we cast the problem of finding shortest-hop routes in discrete networks into the DP framework, and in Section IV-C, using the work of [14], we show how the problem of finding shortest-hop routes in massively dense continuous networks can also be cast into the DP framework. We now take a closer look at the similarities of the two DP formulations.

Note that the fundamental DP equation (4) can be cast in the following form:

$$J_k(x_k) = \min_{x \in U(x_k)} [J_{k+1}(x) + \text{cost}(x, x_k)],$$

$$k = 0, 1, 2, \cdots, N - 2 \quad (9)$$

where $U(x_k)$ represents the set of states reachable by $x_k$.

Equation (9) must be compared to eqs. (5) and (7), which represent the solutions achieved through DP for shortest path routing in the discrete and continuous cases respectively. Note that formally the above equations are the same. In (9) $J_k(x_k)$ represents the minimum of the cumulative cost required to reach the final state starting from state $x_k$ in $(N - k)$ stages. Analogously, in (5) the term $d_k(x^*)$ represents the minimum of the cumulative communication cost required to transfer a packet from node $x^*$ to the final destination $D$ in $(N - k)$ hops. Finally, in (7) the term $S(x, y)$ is the minimum of the optical path length from the point of coordinates $(x, y)$ to the destination. Furthermore, $\text{cost}(x_k, x)$ represents the additional cost associated with the transition from state $x_k$ to state $x$; $d(x, x^*)$ represents the communication cost of a transmission on the link between node $x^*$ and $x$, and $n(x - \sin \theta \cdot \Delta s, y - \cos \theta \cdot \Delta s) \cdot \Delta s$ represents the optical path length between points $(x - \sin \theta \cdot \Delta s, y - \cos \theta \cdot \Delta s)$ and $(x, y)$.

Observe, that there is an evident analogy between minimizing the optical path length and the communication cost if we replace the refraction index $n(x, y)$ with the communication cost of a link between two nodes. The above analogies are summarized in Table IV-D.

As a consequence of the above analogies, the theoretical results achieved in geometrical optics can be utilized to gain insights into the problem of optimal routing in networking in massively dense networks [2].

## V. Conclusions

In this work, we study the problem of routing in large wireless networks by analogy to Optics, building on the previous work in [1], [2], [3]. In particular, we discuss the practical issue of how the nodes in the network can discover, in a distributed manner, the optimal routes, or routes that are close to the optimal. In addition, we present an interpretation of this type of optical routing in terms of Dynamic Programming, using previous work in [14]. As many traditional routing protocols are also based on Dynamic Programming, this interpretation highlights the fundamental connection of optical routing with traditional types of routing.

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**References**


