Mobile Geometric Graphs: Detection, Isolation and Percolation

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Based on joint works with

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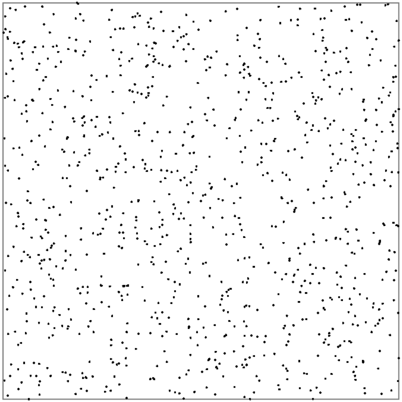
Random Geometric Graph (Boolean Model)

Nodes: Poisson point process in $\mathbb{R}^d$, intensity λ

Edges: $\exists\ (u, v) \iff d(u, v) \leq r$ (*intersecting balls of radius $r/2$*)



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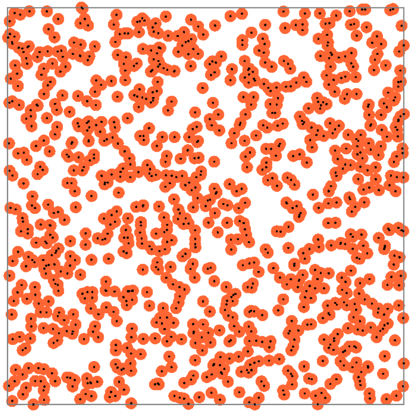

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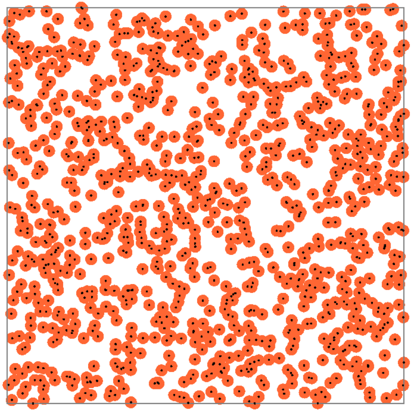

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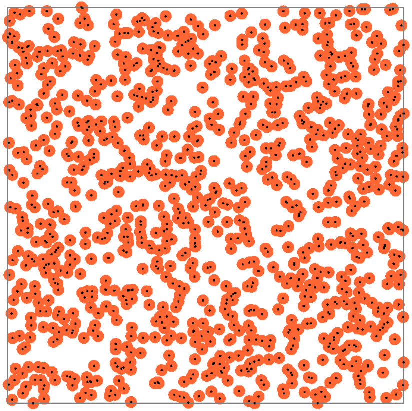

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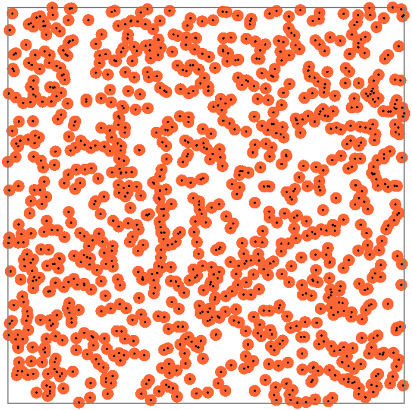


time s

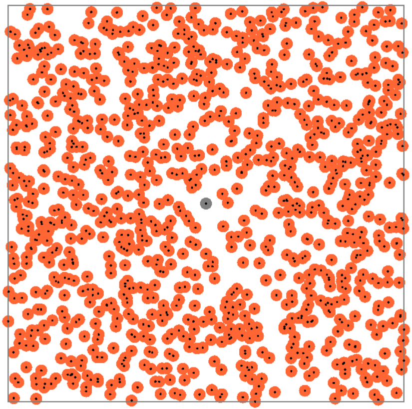
Detection in a mobile geometric graph

Target particle initially at origin

T_{det} = 1st time some node within distance r of target

Want to study $\mathbb{P}(T_{\mathrm{det}} > t)$

$\mathbb{P}(\text{target not detected at fixed } s) = \mathbb{P}(T_{\mathrm{det}} > 0) = e^{-\lambda\pi r^2}$



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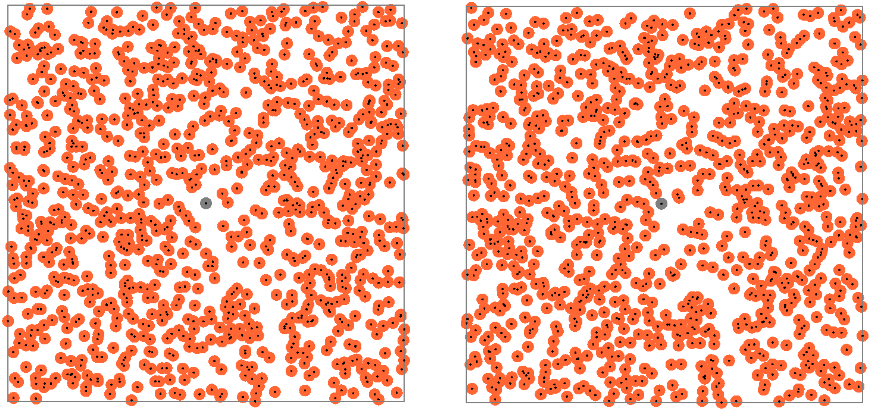

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Detection of a non-mobile target

Lemma (Classical result of stochastic geometry)

Let ξ be a standard Brownian motion and $W_r(t) = \cup_{s \le t} \mathcal{B}(\xi(s), r)$, the Wiener sausage up to time t. Then

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$$\mathbb{E}[\mathrm{vol}(W_r(t))] = \begin{cases} 2\pi \frac{t}{\log t}(1+o(1)), & d=2 \\ c_d r^{d-2} t(1+o(1)), & d \geq 3 \end{cases}$$

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In dimension 3 and above as $t \to \infty$

$$\mathbb{P}(T^f_{\mathrm{det}} > t) \leq \exp\left(-\lambda\alpha(d) c_d r^{d-2} t(1+o(1))\right).$$

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The **symmetric rearrangement** of A, denoted A^*, is a ball centered at the origin with $\mathrm{vol}(A^*) = \mathrm{vol}(A)$.

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Theorem (Special case of Brascamp, Lieb, Luttinger (1974))

Let $A_1, \ldots, A_n \subset \mathbb{R}^d$ of finite volume and $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ a nonincreasing function of distance. Then

$$\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{0 \le i \le n} \mathbb{1}(x_i \in A_i) \prod_{1 \le i \le n} \psi(x_{i-1}, x_i)\, dx_0 \ldots dx_n$$

$$\le \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{0 \le i \le n} \mathbb{1}(x_i \in A_i^*) \prod_{1 \le i \le n} \psi(x_{i-1}, x_i)\, dx_0 \ldots dx_n.$$

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Theorem (Peres, S.)

Let $d \geq 1$ and $f : \mathbb{R}_+ \to \mathbb{R}^d$ be a deterministic motion of the target. Then for all t we have

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Note that f does not need to be continuous.

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Let $(\xi(s))_{s\geq 0}$ be a standard Brownian motion in $d \geq 1$ dimensions and let $(D_s)_{s\geq 0}$ be open sets in $\mathbb{R}^d$ with $\mathrm{vol}(D_s) = c$ for all s. Then for all t we have that

$$\mathbb{E}\left[\mathrm{vol}\left(\cup_{s\leq t}\left(\xi(s) + D_s\right)\right)\right] \geq \mathbb{E}\left[\mathrm{vol}\left(\cup_{s\leq t}\mathcal{B}(\xi(s), r)\right)\right],$$

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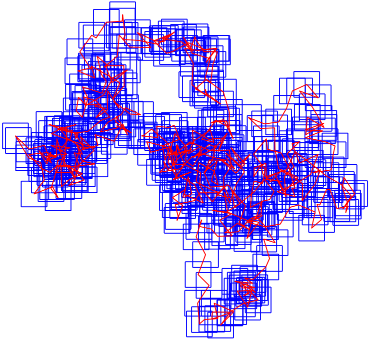
In particular this gives that the expected volume of the Wiener sausage with squares is bigger than the expected volume with balls.

Squares vs disks



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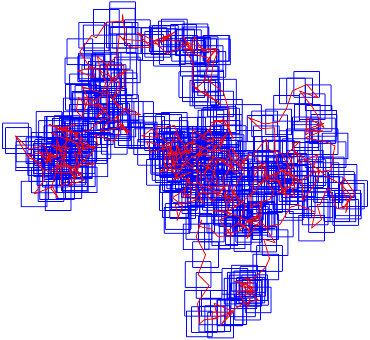

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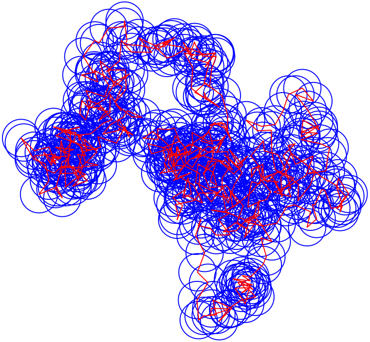

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A connection to capacity

Spitzer and Whitman(1964) proved that in $d \geq 3$, if $A \subset \mathbb{R}^d$ is an open set with finite volume, then

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In $d \geq 3$ among all open sets of fixed volume, the ball has the smallest Newtonian capacity.

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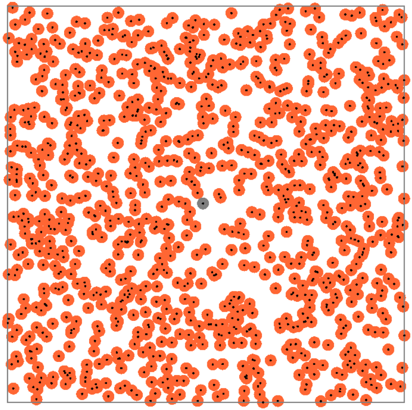

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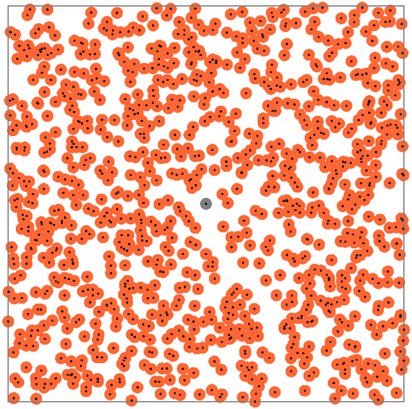

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Define

$$\Psi_d(t) = \begin{cases} \sqrt{t}, & \text{for } d = 1 \\ \log t, & \text{for } d = 2 \\ 1, & \text{for } d \geq 3. \end{cases} \tag{1}$$

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Theorem (Peres, S., Stauffer (2011))

For all $d \geq 1$ as $t \to \infty$

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Lower bound matches upper bound in $d \geq 3$ and up to logarithmic factors in the exponent in $d = 2$.

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Proof uses rearrangement inequalities of Brascamp, Lieb, Luttinger ('74) and a new decoupling idea.

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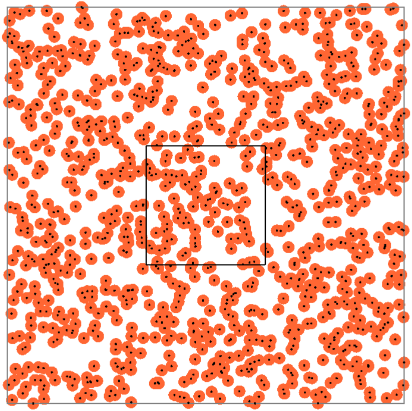
Q_R = cube of side length R

$T_{\rm cov}(Q_R)$ = 1st time all points of Q_R have been detected

Open problem proposed in Konstantopoulos'09.



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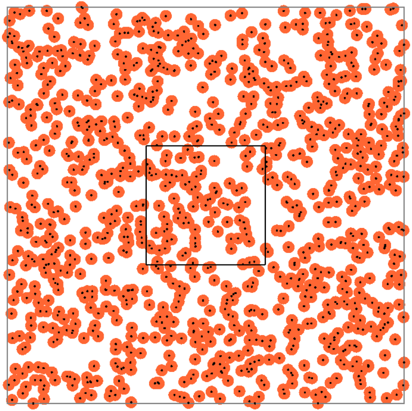

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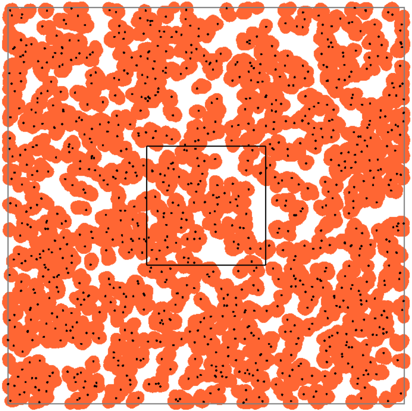

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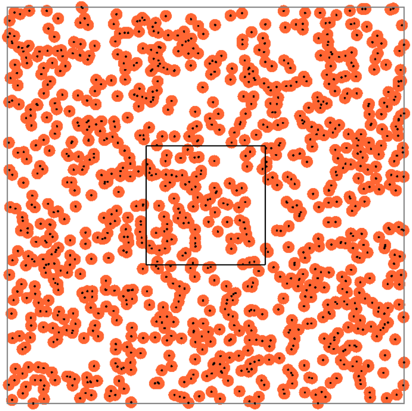
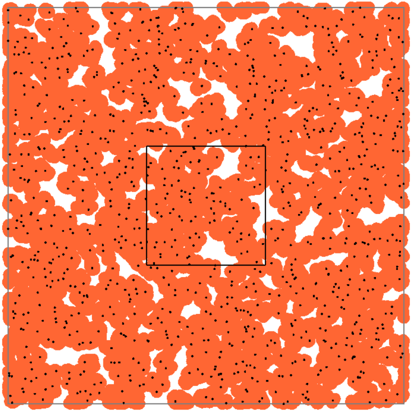
Q_R = cube of side length R

$T_{\text{cov}}(Q_R)$ = 1st time all points of Q_R have been detected

Open problem proposed in Konstantopoulos'09.



time 0



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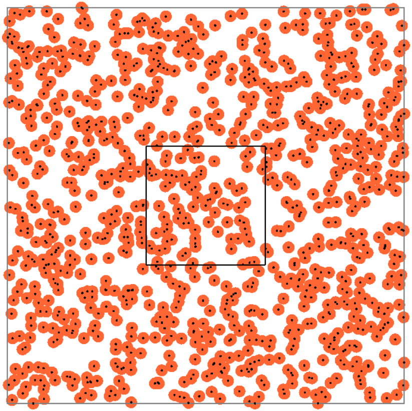

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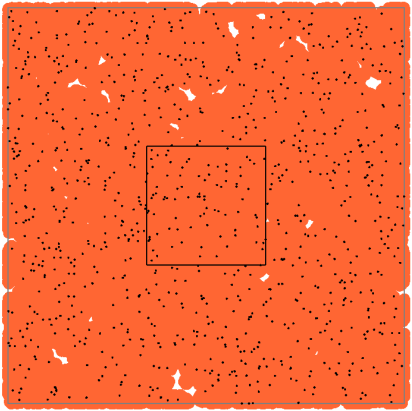

time $T_{\text{cov}}(Q_R)$

Coverage

Theorem (Peres, Sinclair, S., Stauffer)

As $R \to \infty$, we have that

$$\mathbb{E}\, T_{\mathrm{cov}}(Q_R) \sim \frac{2}{2\pi\lambda} \log R \log\log R \quad \textit{and} \quad \frac{T_{\mathrm{cov}}(Q_R)}{\mathbb{E}\, T_{\mathrm{cov}}(Q_R)} \to 1 \textit{ in probability}$$

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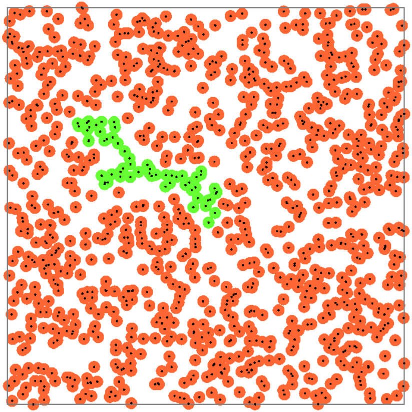
Theorem ((General result), Peres, Sinclair, S., Stauffer)

For a set A and $R > 0$, let $RA = \{Ra : a \in A\}$. If A has Minkowski dimension α, then as $R \to \infty$

$$\mathbb{E}\, T_{\mathrm{cov}}(RA) \sim \frac{\alpha}{2\pi\lambda} \log R \log\log R \quad \textit{and} \quad \frac{T_{\mathrm{cov}}(RA)}{\mathbb{E}\, T_{\mathrm{cov}}(RA)} \to 1 \textit{ in probability}$$

Percolation on Mobile Geometric Graph

$\exists \lambda_c$ s.t. $\lambda > \lambda_c \Rightarrow$ *a.s.* $\exists$ infinite component at fixed time *s*



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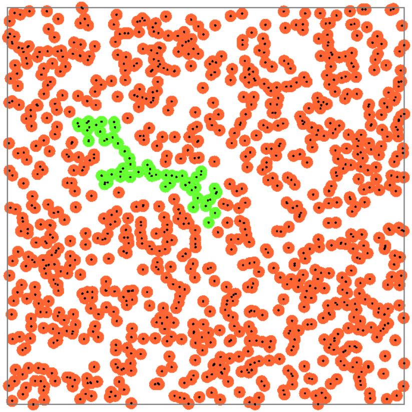

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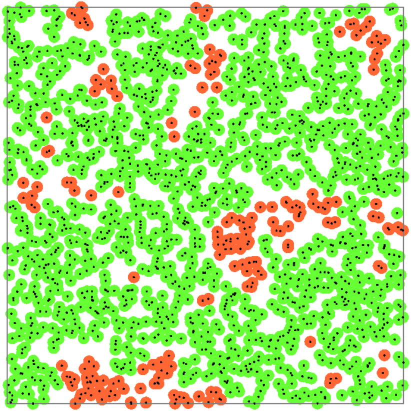
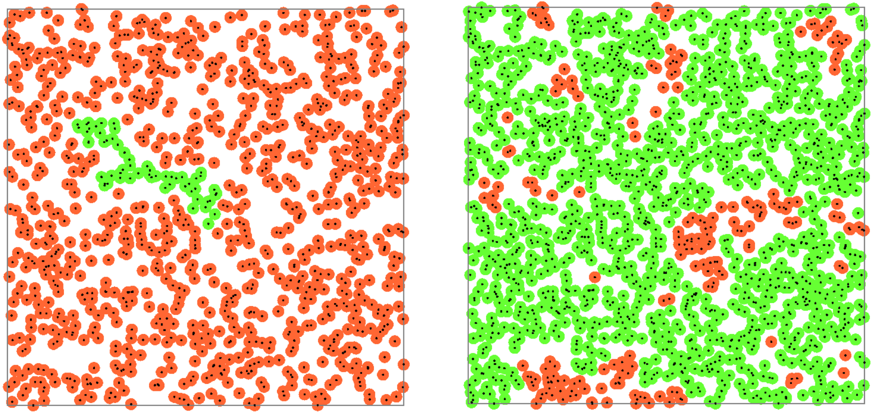


$\lambda > \lambda_c$

Percolation on Mobile Geometric Graph

$\lambda > \lambda_c \Rightarrow$ *a.s.* $\exists$ infinite component for every s (van den Berg, Meester, White'97)



$\lambda < \lambda_c$ $\lambda > \lambda_c$

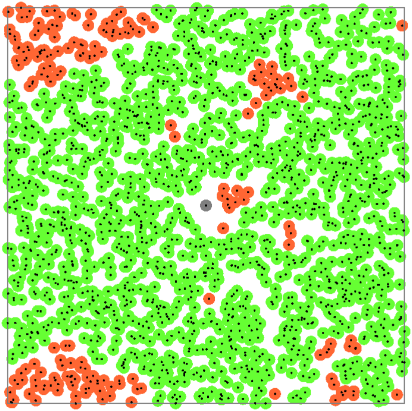
Percolation

Target particle initially at origin

We assume $\lambda > \lambda_c$

T_{perc} = 1st time target belongs to infinite component

Want to study $\mathbb{P}(T_{\text{perc}} > t)$



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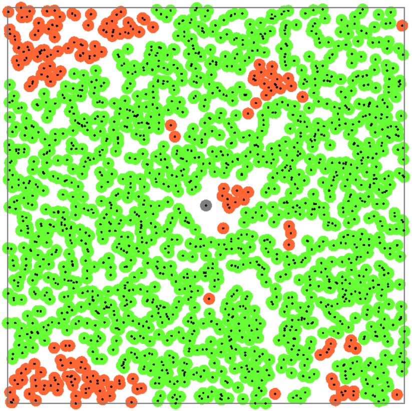

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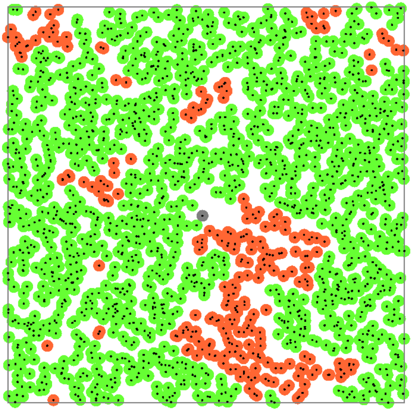

time T_{perc}

Percolation

Lower bound in discrete time via FKG (extends to continuous time):

$\mathbb{P}(\text{target} \notin \text{infinite component at time } s) = \mathbb{P}(T_{\text{perc}} > 0) = e^{-c}$

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Lower bound via detection:

$\mathbb{P}(T_{\mathrm{perc}} > t) \geq \mathbb{P}(T_{\mathrm{det}} > t) \geq e^{-c'' t/\log t}$

Percolation

Upper bound:

$\mathbb{P}(T_{\mathrm{perc}} > t) \leq \exp\left(-c\sqrt{t}\right)$ (Sinclair, Stauffer 2010)

Percolation

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Theorem (Peres, Sinclair, S., Stauffer (2010))

If $\lambda > \lambda_{\mathrm{c}}$, then $\exists$ constant c s.t.

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Proof uses coupling and multiscale analysis.

Y. Peres, A. Sinclair, P. Sousi, and A. Stauffer.
Mobile Geometric Graphs: Detection, Coverage and Percolation.
Prob. Theory and Related Fields. 156 (2013), no. 1-2, 273–305.

Y. Peres and P. Sousi.
An isoperimetric inequality for the Wiener sausage.
Geom. Funct. Anal., 22(4):1000–1014, 2012.

Yuval Peres, Perla Sousi, and Alexandre Stauffer.
The isolation time of Poisson Brownian motions.
ALEA Lat. Am. J. Probab. Math. Stat., 10(2):813–829, 2013.