Statistical Physics on Sparse Random Graphs: Mathematical Perspective

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Outline

1. Motivation: why should we care about it?
2. Interesting phenomena
3. Uniformly sparse, locally tree-like graphs
4. Interpolation techniques
5. Specific mathematical results
   - Ferromagnetic Ising model
   - Spin glasses at high temperature
   - Potts and hard-core models: $\mathbf{T}_d$-like graphs
   - Anti-ferromagnetic binary models: complexity issues
6. Conclusion

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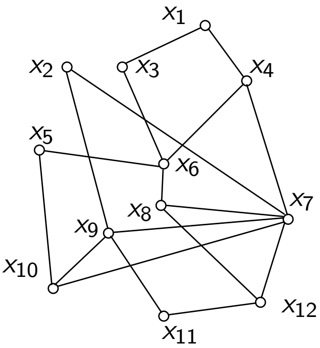
Why should we care?

'Standard model'



$G = (V, E)$, $V = [n]$, $\underline{x} = (x_1, \dots, x_n)$, $x_i \in \mathcal{X}$ (finite set).

$$\mu(\underline{x}) = \frac{1}{Z} \prod_{(ij) \in E} \psi_{ij}(x_i, x_j)\,.$$

'Standard model' (assumptions)

1. G has bounded degree, on average.

2. G has girth larger than 2ℓ with $\ell = \ell(n) \to \infty$ (apart from $o(n)$ vertices).

3. permissible $0 \le \psi_{ij}(x_i, x_j) \le \psi_{\max} < \infty$. For each i exists x_i^{p} s.t. $0 < \psi_{\min} \le \psi_{ij}(x_i^{\mathrm{p}}, x_j)$.

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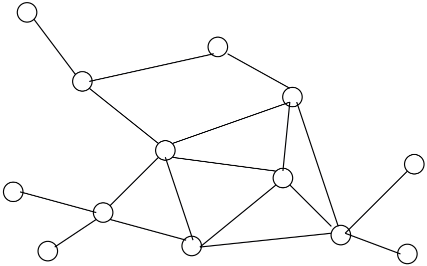
Example 1: q-coloring



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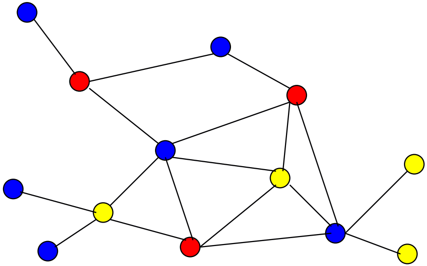

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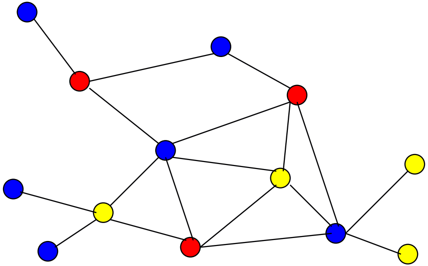
Uniform measure over proper colorings



$$\mu(\underline{x}) = \frac{1}{Z} \prod_{(i,j)\in E} \psi(x_i, x_j)\,, \qquad \psi(x, y) = \mathbb{I}(x \neq y)\,.$$

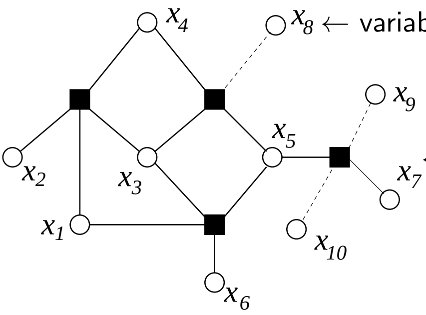
Example 2: k-satisfiability

n variables: $\underline{x} = (x_1, x_2, \ldots, x_n)$, $x_i \in \{0, 1\}$

m k-clauses

$$(x_1 \vee \overline{x_5} \vee x_7) \wedge (x_5 \vee x_8 \vee \overline{x_9}) \wedge \cdots \wedge (\overline{x_{66}} \vee \overline{x_{21}} \vee \overline{x_{32}})$$

Uniform measure over solutions



$\leftarrow$ variables $x_i \in \{0,1\}$

$\leftarrow$ clauses, e.g. $(x_5 \vee x_7 \vee \overline{x_9} \vee \overline{x_{10}})$

$$F = \cdots \wedge \underbrace{(x_{i_1(a)} \vee \overline{x}_{i_2(a)} \vee \cdots \vee x_{i_k(a)})}_{a\text{-th clause}} \wedge \cdots$$

$$\mu(\underline{x}) = \frac{1}{Z} \prod_{a=1}^{m} \psi_a(x_{i_1(a)}, \ldots, x_{i_k(a)})$$

Many other examples

Communications (LDPC; XORSAT).

Artificial intelligence (Bayesian networks; Graphical models).

Statistics (Compressed sensing).

...

Interesting phenomena

1. 'Exact' predictions: free energy density

Graph sequences $G_n = ([n], E_n)$

$$Z_n \equiv \sum_{\underline{x}} \prod_{(ij)\in E_n} \psi_{ij}(x_i, x_j)$$

Much interest in the (asymptotic) **free energy density**

$$\phi \equiv \lim_{n\to\infty} n^{-1} \log Z_n \quad \text{(if limit exists);}$$

exact evaluation of ϕ yields substantial information on the thermodynamic limit of the system (phase transitions, etc.)

['Done' by Cavity/Replica methods]

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2. Mean field equations

'Set of $O(n)$ non-linear equations that determine local marginals in the large system limit'

2. Mean field equations

Bethe-Peierls equations (replica symmetric cavity method):

$\mu_{i\to j}(\,\cdot\,) \equiv$ Marginal of x_i when replacing $\psi_{ij}(x_i, x_j)$ by 1

$$\mu_{i\to j}(x_i) \approx \frac{1}{z_{i\to j}} \prod_{l\in\partial i\setminus j} \sum_{x_l} \psi_{il}(x_i, x_l)\mu_{l\to i}(x_l)$$

General philosophy: approximate local marginals of $\mu(\,\cdot\,)$ in terms of measures on trees.

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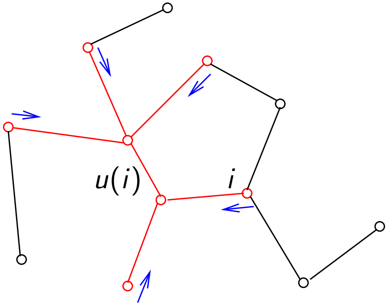
2. Bethe-Peierls approximation



$$F = (U, E_U) \subseteq G,\ \mathrm{diam}(F) \le 2\ell, \text{ such that } \partial i \in U \text{ or } \partial i \cap U = \{u(i)\}$$

$$\mu_U(\underline{x}_U) \approx \nu_U(\underline{x}_U) = \frac{1}{Z_U} \prod_{(i,j)\in E_U} \psi_{ij}(x_i, x_j) \prod_{i\in\partial U} \nu_{i\to u(i)}(x_i)\,.$$

Questions

$\{\nu_{i\to j}(\,\cdot\,)\}$ → 'set of messages' (aka cavity fields)

1. Is $\mu(\,\cdot\,)$ well-approximated by some $\{\nu_{i\to j}(\,\cdot\,)\}$?
2. How to find a good set of messages $\{\nu_{i\to j}(\,\cdot\,)\}$?

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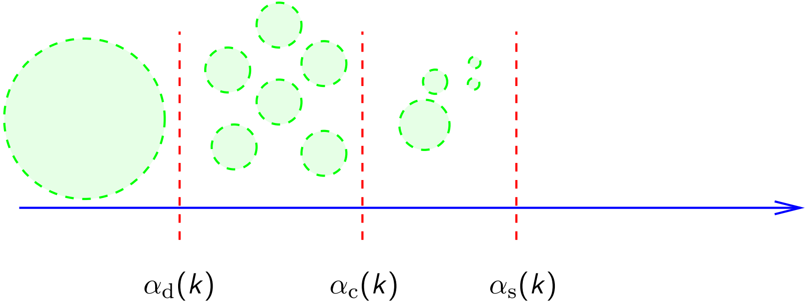
3. 'Dynamical' phase transition

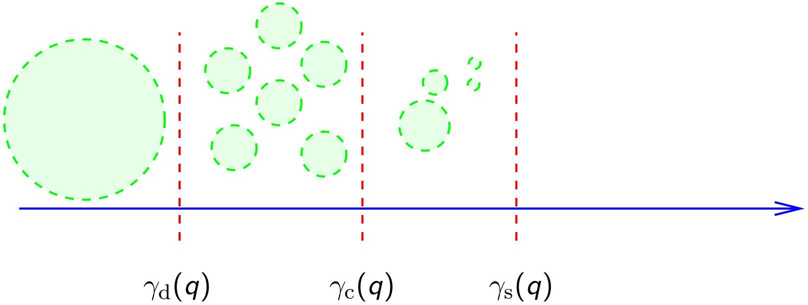
'The free energy density is analytic but the measure μ splits into lumps'

Example: *k*-satisfiability, the space of solutions



$\alpha = m/n$ fixed, $n \to \infty$.
[Biroli,Monasson,Weigt 00, Mézard,Parisi,Zecchina 02, Krzákala et al 07]

Example: q-COL, the space of solutions



[Edges taken independently with probability γ/n each, γ fixed, $n \to \infty$]
[same references + Achlioptas,Ricci 06]

4. 'Non-self averaging'

$\underline{x}^{(1)}, \underline{x}^{(2)}$ independent configurations, same disorder (replicas)

$d(\underline{x}^{(1)}, \underline{x}^{(2)})$ Hamming distance

$\mu(d(\underline{x}^{(1)}, \underline{x}^{(2)}) > n\delta) \to$ non-degenerate random variable

[$\sim$ SK model]

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Uniformly sparse, locally tree-like graphs

Locally tree-like graphs: regular limit

Consider graphs that are **sparse** ($|E_n| \asymp |V_n|$) and **locally tree-like** — *the local neighborhood of a uniformly random vertex converges in distribution to a (random) rooted tree*

the random d-regular graph $\mathscr{G}_{n,d}$ converges (locally) to the infinite d-regular tree $\mathbf{T}_d$, denoted $\mathscr{G}_{n,d} \rightsquigarrow \mathbf{T}_d$

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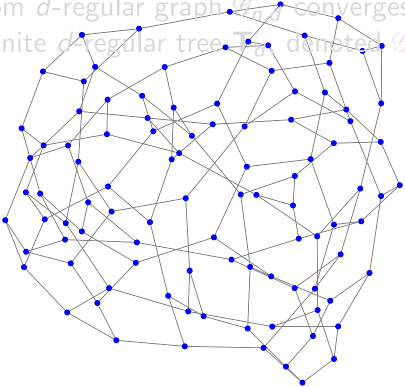

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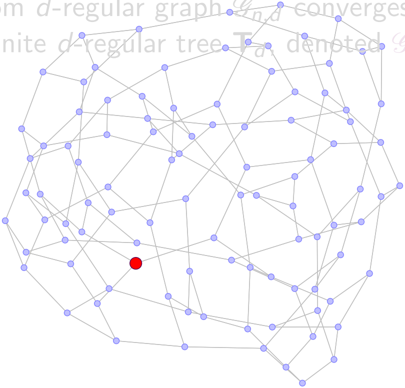

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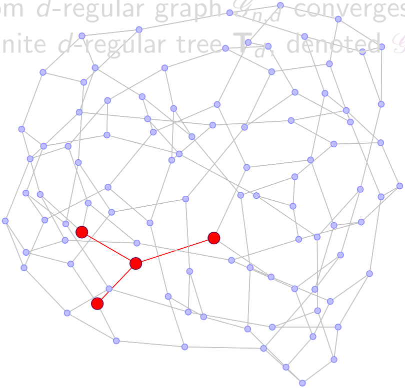

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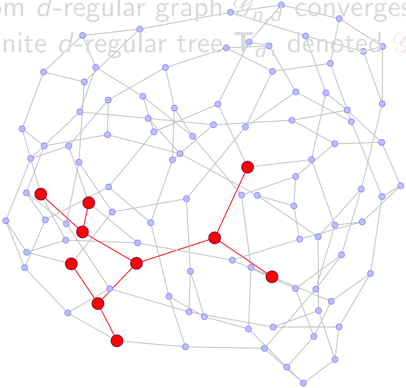

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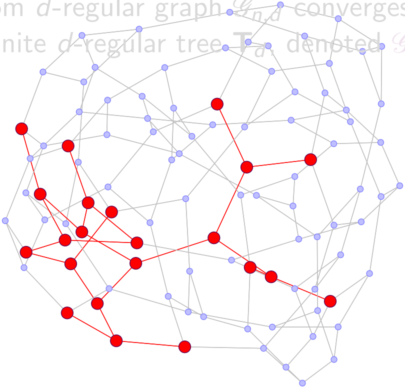

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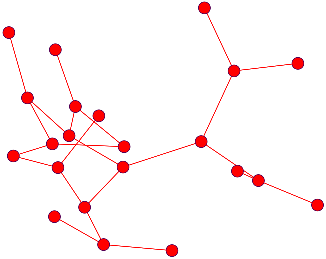

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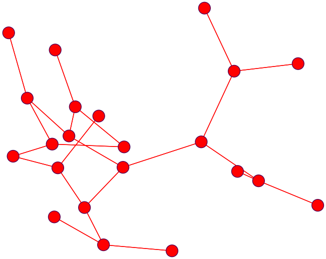
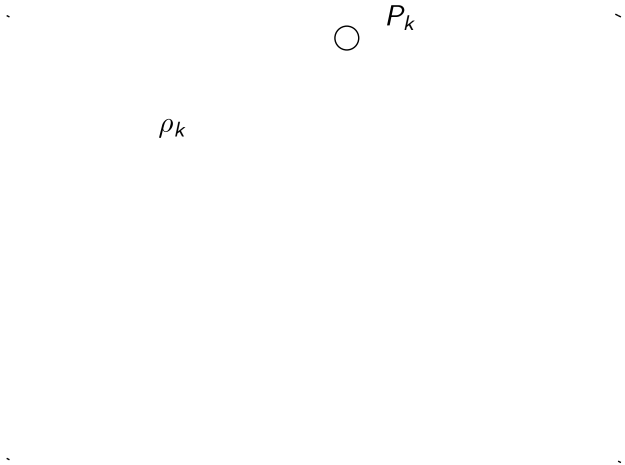


UGW $T(P, \rho, t)$



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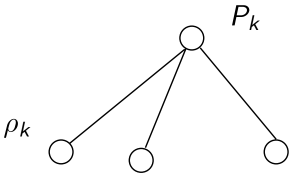

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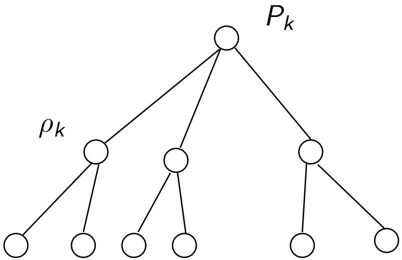

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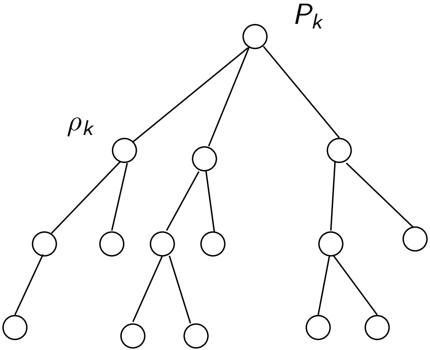

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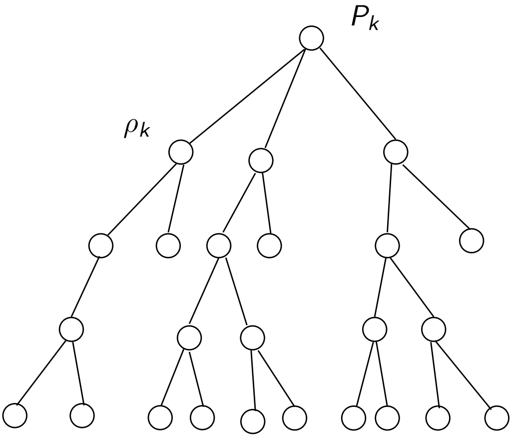

Uniformly sparse, locally tree-like graphs: GW limit

The *Unimodular* Galton-Watson tree:

$P \equiv \{P_k\}_{k\geq 0}$	Degree distribution, law of L (of finite mean $\overline{P}$)
$\rho \equiv \{\frac{k}{\overline{P}} P_k\}_{k\geq 0}$	Law of K (degree of uniform edge)
$\mathsf{T}(P,\rho,t)$	t-generations UGW tree (root degree P, else ρ)
$\mathsf{B}_i(t)$	Ball of radius t in G_n centered at node i

Definition

$\{G_n\}$ *converges locally* to $\mathsf{T}(P,\rho,\infty)$ if for uniformly random $I \in [n]$ and fixed t, law of $\mathsf{B}_I(t)$ converges as $n \to \infty$ to $\mathsf{T}(P,\rho,t)$.
$\{G_n\}$ *uniformly sparse* if $\{|\partial I|\}$ is uniformly integrable.

see Benjamini–Schramm '00, Aldous–Lyons '06

If $G_n \rightsquigarrow \mathsf{T}$ a random tree of law Q, then Q must be *unimodular*:

$$\int \sum_{v \in V} f(G, o, v) \mathrm{d}Q([G, o]) = \int \sum_{v \in V} f(G, v, o) \mathrm{d}Q([G, o]) \,, \; \forall f(\cdot, \cdot, \cdot) \,.$$

Choosing rooted graph $[G, o]$ according to the measure Q biased by the degree of the root, then performing SRW by moving the root uniformly among all adjacent vertices, yields a reversible MC.

[Aldous–Lyons '06]

Unimodularity extends to models μ_n on G_n that converge locally to a model ν on T (for any fixed t, joint law of $\mathsf{B}_l(t)$ and $\underline{x}_{\mathsf{B}_l(t)}$ converges to $\mathsf{T}(t)$ with $\underline{x}_{\mathsf{T}(t)}$ of law $\nu_{\mathsf{T}(t)}$ on it).

Tree-like graphs: motivations

The definition encompasses natural random graph ensembles such as sparse Erdős–Rényi graphs or random regular graphs; many natural randomized computational problems are also described by a locally tree-like constraint structure (random k-SAT, q-coloring)

Trees are amenable to *exact analysis* (recursive equations) — physicists predict many exact asymptotic results for the setting $G_n \rightsquigarrow T$, based on comparisons between models on G_n and on T

The **Bethe replica symmetric** prediction for ϕ in graphs $G_n \rightsquigarrow T$: $\phi = \Phi^{\text{Bethe}}$ explicit which *depends only on the limit tree T*

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Bethe Gibbs measures

On $G_n \rightsquigarrow T$, $\Phi^{\text{Bethe}} = \Phi(\nu^\star)$, with Φ an explicit function, and $\nu^\star$ a certain infinite-volume Gibbs measure on T — represents proposed local limit of the models on G_n, "$\mu_n \rightsquigarrow \nu^\star$"

The replica symmetric prediction assumes special structure for $\nu^\star$: marginal on any finite subgraph $U \subset T$ given by

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Tree-like graphs: examples

Random regular graph $\mathscr{G}_{n,d} \rightsquigarrow$ regular tree $\mathbf{T}_d$

Sparse Erdős–Rényi graph $G_{n,d/n} \rightsquigarrow \mathrm{T}(P,\rho,\infty)$
with $P = \rho$ both Poisson(d)

Graphs of fixed degree distribution $\rightsquigarrow$ GW trees

Depth-t subtree $\mathbf{T}_d(t) \rightsquigarrow$ random d-canopy tree, **not** $\mathbf{T}_d$
see Aizenman–Warzel '06, D.–Montanari '09

$G_n \rightsquigarrow T$ compares the rooted tree T with G_n rooted at a (uniformly) random vertex — $\mathbf{T}_d(t) \not\rightsquigarrow \mathbf{T}_d$ because a random vertex in $\mathbf{T}_d(t)$ is likely to be at or near the leaves

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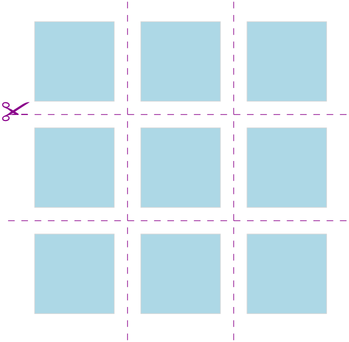

Classical physics focused on finite exhaustions of finite-dimensional lattice graphs, where existence of ϕ can usually be deduced by subadditivity — deleting a negligible fraction of edges decomposes the graph into blocks (amenable)

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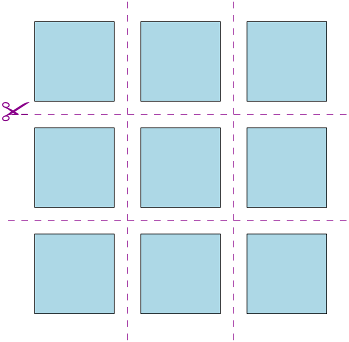

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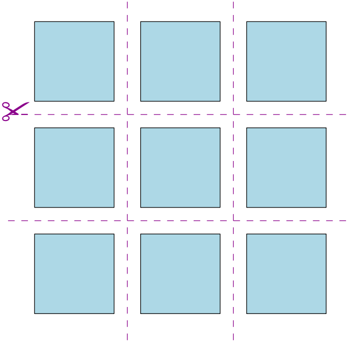

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Interpolation techniques

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Early breakthrough in *Guerra–Toninelli interpolation scheme* (2002), a sub-additive approach proving existence of ϕ for SK spin glasses

Scheme has been significantly developed in subsequent work, extending to cover some sparse graphical models

Guerra '02, Franz–Leone '03, Franz–Leone–Toninelli '03, Panchenko-Talagrand '04, Bayati-Gamarnik-Tetali '09, Gamarnik '12, Salez '14

(in some cases with quantitative bounds on ϕ)

However, this interpolation method has only been successfully applied in somewhat special random graph ensembles

(Erdős–Rényi, or uniformly random given degree sequence)

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Among ferromagnetic models, simplest is **Ising with $\beta \geq 0$**

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Specific mathematical results

Ferromagnetic Ising model

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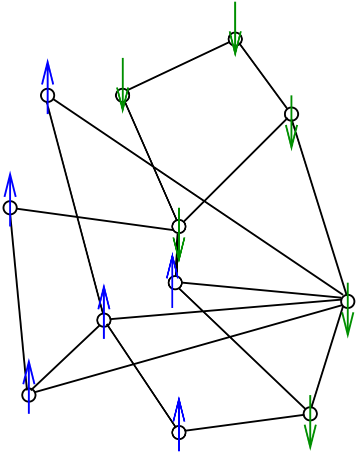

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[in sparse random graphs: Johnston–Plechác 98, Leone et al 04]

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any limiting tree & variational formulation for $\mathbf{\Phi}^{\mathrm{Bethe}}$ DMS11]

0. Take $B > 0$.

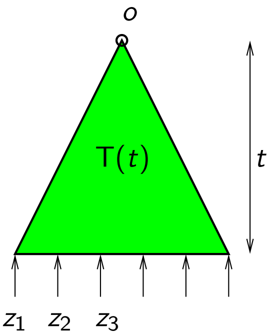
1. Reduce to expectations of local quantities

$$\frac{\mathrm{d}}{\mathrm{d}B} \log Z_n(\beta, B) = \sum_{i=1}^{n} \langle x_i \rangle_n$$

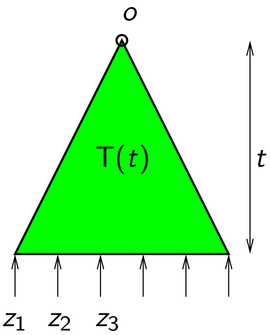
($\langle \cdot \rangle_n$ denote expectation under Ising on G_n).

2. Prove convergence of local expectations to tree values.

2. Convergence to tree values



T	infinite tree with max degree $k_{\max}$
$\mathrm{T}(t)$	first t generations
$\mu^{t,z}(\cdot)$	Ising model on $\mathrm{T}(t)$ boundary condition z
$\mu_o^{t,z}$	root spin expectation



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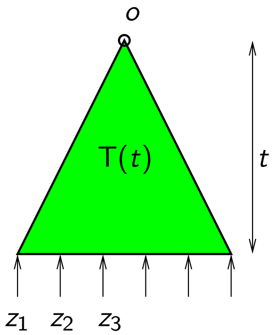

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Uniform (Gibbs measure uniqueness)

$$|\mu_o^{t,z(1)} - \mu_o^{t,z(2)}| \le |\mu_o^{t,+} - \mu_o^{t,-}| \to 0 \,.$$

Easier

True only at high temperature ($\beta = O(1/k_{\max})$)

Decorrelation

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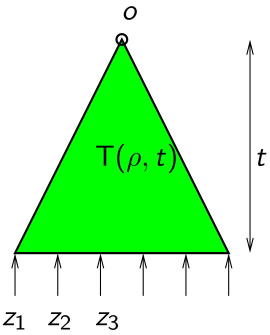
Easier

True only at high temperature ($\beta = O(1/k_{\max})$)

Why non-uniform control? Phase transition. . .

For $\beta > \beta_{\rm c} \equiv \operatorname{atanh}(1/\overline{\rho})$

$$\lim_{B\to 0+} \lim_{n\to\infty} \mathbb{E}\langle x_I \rangle_n = - \lim_{B\to 0-} \lim_{n\to\infty} \mathbb{E}\langle x_I \rangle_n > 0$$



$$z = (+1, +1, \dots, +1) \quad \Rightarrow \quad \lim_{t\to\infty} \langle x_o \rangle_t > 0$$

$$z = (-1, -1, \dots, -1) \quad \Rightarrow \quad \lim_{t\to\infty} \langle x_o \rangle_t < 0$$

Decorrelation

Non uniform

$$|\mu_o^{t,+} - \mu_o^{t,\mathrm{free}}| \to 0 \, .$$

Any temperature

Trick

$$0 \leq \mu_o^{t,+} - \mu_o^{t,\mathrm{free}} \leq \epsilon\{\mu_o^{t,\mathrm{free}} - \mu_o^{t-1,\mathrm{free}}\} \to 0$$

($\mu_o^{t,\mathrm{free}}$ monotone by Griffiths)

[Ising specific, but strategy extends to Potts, Hard-core models]

Phase transition: Ising model

On a variety of large graphs $G_n = (V_n, E_n)$, for $\beta > \beta_c$ and $B = 0$, the Ising measure $\mu_n(\cdot)$ decomposes into convex combination of well separated simpler components.

Mathematically proved for grids [Aizenman '80; Dobrushin, Shlosman '85; Georgii, Higuchi '00; Bodineau '06], and for the complete graph [Ellis, Newman '78].

Proved in [Montanari, Mossel, Sly '09] for regular graph sequences $G_n \rightsquigarrow \mathbf{T}_d$; Extended to any $G_n \rightsquigarrow \mathsf{T}$ with $\beta \mapsto \mathbb{E}[\sum_{i \in \partial o} \nu_{+,\mathsf{T}}^{\beta,0}(x_o x_i)]$ continuous [Basak–D. '12]

Results of [MMS09]

For $B = 0$ any $\beta \geq 0$,

$$\mu_n(\cdot) \to \frac{1}{2}\nu_{+,\mathsf{T}_d}(\cdot) + \frac{1}{2}\nu_{-,\mathsf{T}_d}(\cdot)\,.$$

$\nu^{\beta,B}_{\pm,\mathsf{T}}$ Ising on T with plus/minus boundary conditions.

Let $\mu_{n,+}$ and $\mu_{n,-}$ denote the Ising measures on G_n conditioned on $\sum_i x_i > 0$ and $\sum_i x_i < 0$, respectively.

For $B = 0$ and any $\beta \geq 0$, subject to an edge-expansion condition

$$\mu_{n,\pm}(\cdot) \to \nu_{\pm,\mathsf{T}_d}(\cdot)\,.$$

($B \neq 0$ not interesting and follows from DM09)

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Edge-expansion condition

Definition

A finite graph $G = (V, E)$ is an $(\delta, 1/2, \lambda)$ *edge-expander* if, any $S \subseteq V$ with $\delta|V| \leq |S| \leq |V|/2$, has $|E(S, S^c)| := |\partial S| \geq \lambda|S|$.

Configuration models are edge-expanders (degrees ≥ 3 needed).

- Example: Consider ℓ identical, disjoint copies of same realization of d-regular graph from $\mathscr{G}_{n,d}$. Condition on $\sum_i x_i > 0$. Then, $\mu_{n\ell,+} \Rightarrow \alpha_\ell \nu_{+,\mathsf{T}_d} + (1 - \alpha_\ell)\nu_{-,\mathsf{T}_d}$, with $\alpha_\ell = 1/2 + O(\ell^{-1/2})$ the probability of sum of the spins being positive in each component.

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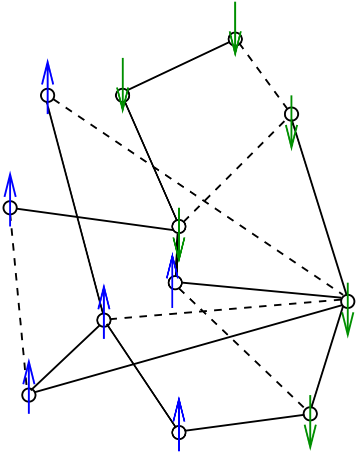
For most i the neighborhood $\mathrm{B}_i(t)$ converges to $\mathbf{T}_d(t)$.
Likely $\mu_{n,\mathrm{B}_i(t)}(\cdot)$ converges to the marginal on neighborhood of the root, for some Ising Gibbs measure on $\mathbf{T}_d$.
For large β there are (uncountably) many such Gibbs measures.
Which one to choose?
Recall the Ising measure:

$$\mu(\underline{x}) = \frac{1}{Z(\beta, B)} \exp\Big\{\beta \sum_{(i,j)\in E} x_i x_j + B \sum_{i\in V} x_i\Big\}.$$

- $\beta = 0 \Rightarrow$ independence (unique measure).
- $\beta = \infty$, and $B = 0 \Rightarrow$ with prob. $1/2$ all spins $+$, and with prob. $1/2$, all are $-$.

Ising spin glasses (high temperature phase)

Ising spin glass



Ising spin glass

$G_n = (V_n \equiv [n], E_n)$ finite, undirected graphs.

$x_i \in \{+1, -1\}$

$$\mu(\underline{x}) = \frac{1}{Z_{n,\mathbf{J}}(\beta, B)} \exp\left\{\beta \sum_{(ij)\in E_n} J_{ij} x_i x_j + B \sum_{i=1}^{n} x_i\right\}$$

$J_{ij} \in \{+1, -1\}$ uniformly random

[Viana–Bray '85]

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Ising spin glass: Free energy density

$$\phi = \lim_{n\to\infty} \phi_{n,\mathbf{J}} \qquad \phi_{n,\mathbf{J}} = \frac{1}{n} \log Z_{n,\mathbf{J}}(\beta, B)\,.$$

Theorem

If $G_n \rightsquigarrow \mathsf{T}(P, \rho, \infty)$ and $\beta < \beta_*(B, P)$, then $\phi = \Phi^{Bethe}(P, \beta, B)$.

$G_n \in \mathscr{G}_{n,d}$

$B = 0$, $\beta_* = \mathrm{atanh}(1/\sqrt{d})$ [Guerra-Toninelli '03]
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[D.-Gerschenfeld-Montanari '14]

$\beta_*(0, P) = \mathrm{atanh}(1/\sqrt{\overline{\rho}})$; $K \sim k_{\rm typ} \gg 1 \Rightarrow \beta_*(B, P) \simeq \frac{f(B)}{\sqrt{k_{\rm typ}}}$ and $0 < f(B) \uparrow \infty$ with B.

Proved by interpolation in phase-space, with non-uniform decorrelation in [DGM14]

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Potts and hard-core models: $\mathbf{T}_d$-like graphs

Potts models

On graph $G = (V, E)$, consider **Potts models**

$$\nu(\underline{x}) = \frac{1}{Z} \prod_{(ij)\in E} e^{\beta \mathbf{1}\{x_i = x_j\}}$$

— distribution on spin configurations $\underline{x} \in [q]^V$, $[q] \equiv \{1, \ldots, q\}$, with **partition function** Z

Interaction strength parametrized by *inverse temperature* β; may add *external field* B towards distinguished spin 1

Natural generalization of **Ising models** $(q = 2)$; coupled with Fortuin–Kasteleyn (FK) random-cluster models for $\beta, B \geq 0$

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$$\nu(\underline{x}) = \frac{1}{Z} \prod_{(ij)\in E} (1 - x_i x_j)$$

— supported on $\underline{x} \in \{0,1\}^V$ encoding **independent sets**

(no neighbors occupied)

Occupied vertices are weighted by a *fugacity* λ

The (approximate) evaluation of $Z(\lambda)$ on given input graphs is a much-studied computational problem.

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Locally regular-tree like graphs

Results obtained partly by interpolation in parameters and partly by other means . . .

mostly in the setting $G_n \rightsquigarrow \mathbf{T}_d$ (regular tree) — the only possibility for a non-random limiting tree (unimodularity) — where the Bethe prediction takes a simpler form

Recall examples of graphs $G_n \rightsquigarrow \mathbf{T}_d$ include
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Results: Free energy

(by interpolation in the parameters)

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Bethe Gibbs measures: Non-uniqueness

A major challenge in verifying the Bethe prediction is the **non-uniqueness** of infinite-volume Gibbs measures

Ising already has non-uniqueness, but the correct measure can be isolated by monotonicity considerations DM09

Situation is qualitatively different in $q \geq 3$ Potts models, where monotonicity considerations do not close the gap

If multiple Gibbs measures, physics prescription is to select *Bethe* Gibbs measure achieving highest value of Φ^{Bethe},
$\Phi^{\text{Bethe}} = \sup\{\Phi(\nu^\star) : \nu^\star \text{ Bethe Gibbs measure}\}$
(restriction to Bethe Gibbs measures is an assumption)

— [DMSS12] prove such prediction holds
(within one non-trivial example).

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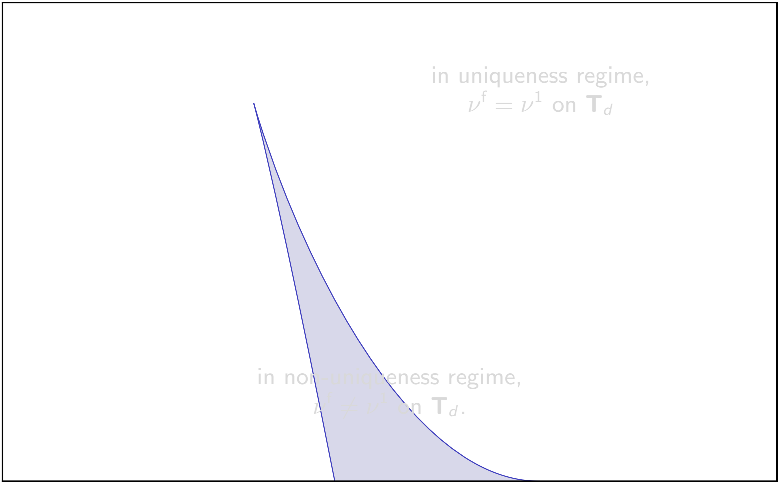
Potts: Non-uniqueness regime

Non-uniqueness regime for Potts on $\mathbf{T}_d$ $(d = 4, q = 30)$



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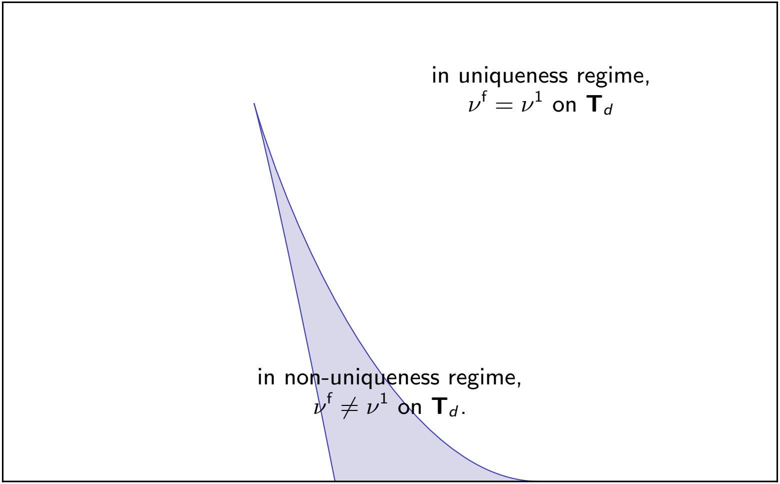

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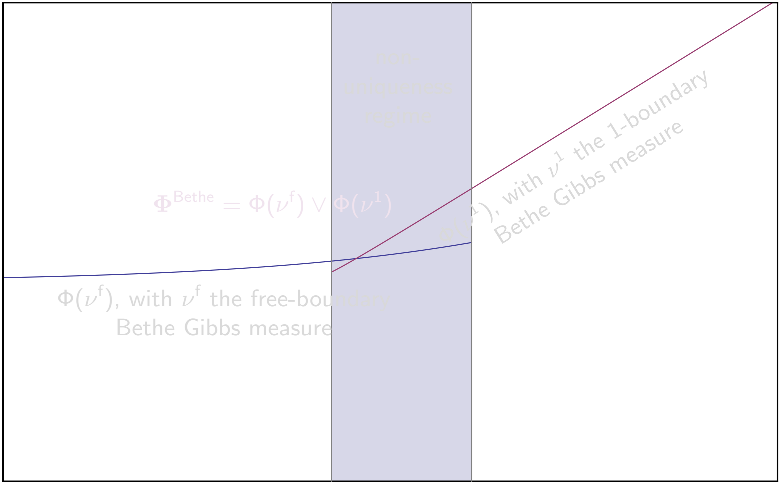
Potts Bethe prediction as function of β $(d = 4,\ q = 30,\ B = 0.05)$



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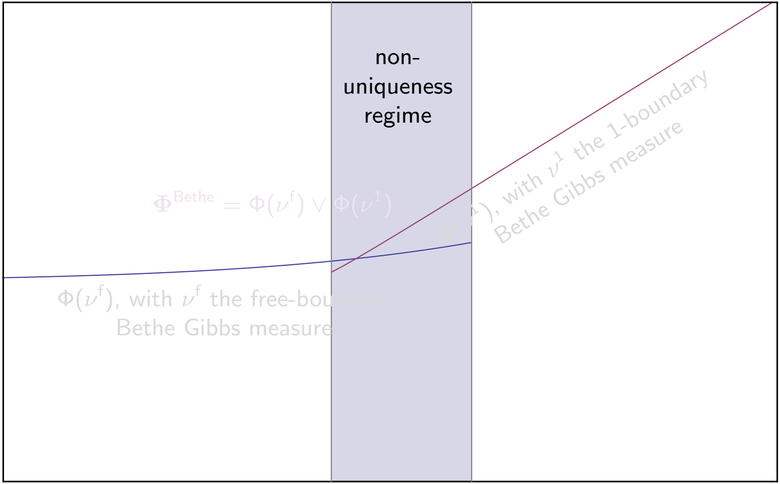

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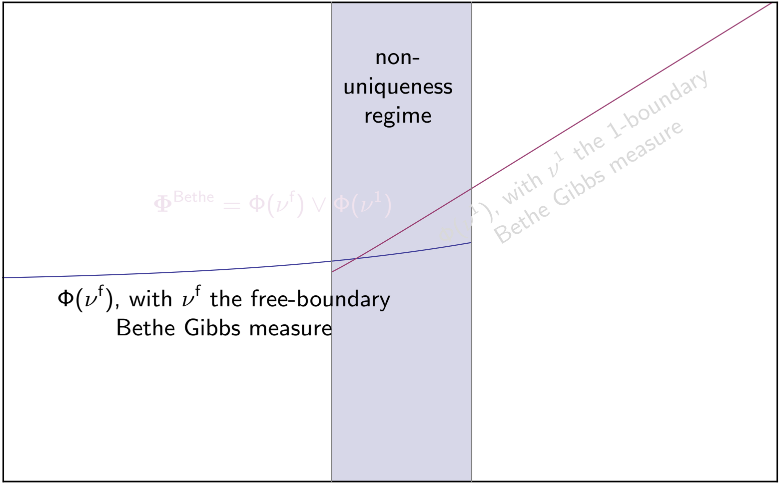

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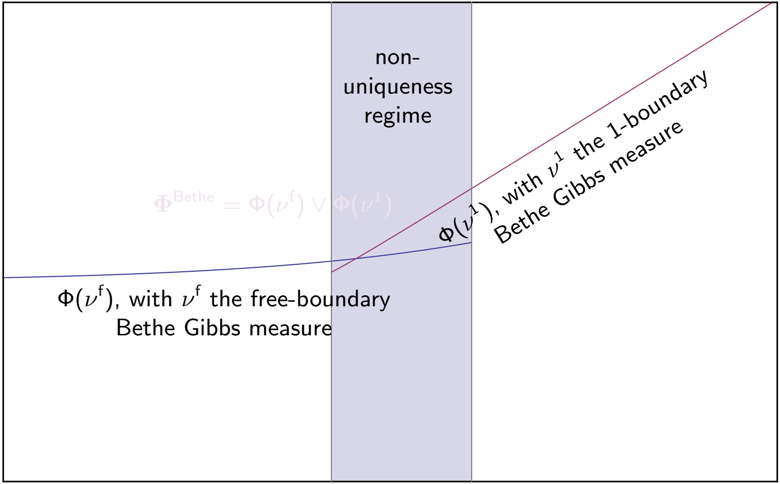

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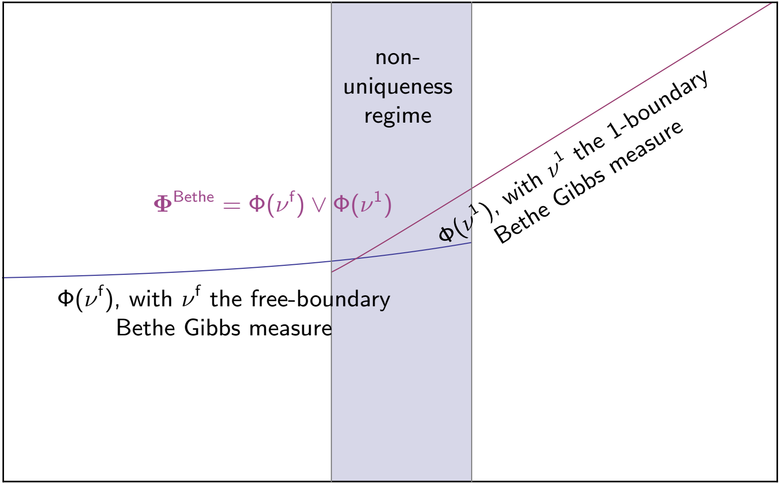

Anti-ferromagnetic binary models: complexity issues

(hard-core and anti-ferromagnetic Ising)

AF binary: Non-uniqueness

Hard-core model on $\mathbf{T}_d$: see e.g. Kelly '85
there is a unique Gibbs measure for any $\lambda \leq \lambda_{\mathsf{u}}(d)$

For $\lambda > \lambda_{\mathsf{u}}(d)$, boundary condition has nonvanishing influence on root — in particular, $\mathbf{T}_d$ exhibits distinct Gibbs measures $\nu^{\text{even}} \neq \nu^{\text{odd}}$ giving maximal preference to even (odd) levels

The AF Ising model similarly has a Gibbs non-uniqueness regime, demarcated by a curve in the (β, B) space

Bipartite Bethe prediction $\Phi^{\text{Bethe}}_{\text{bip}}$ allows *semi-translation-invariant* Bethe Gibbs measures like $\nu^{\text{even}}, \nu^{\text{odd}}$ while standard Bethe prediction Φ^{Bethe} requires translation invariance

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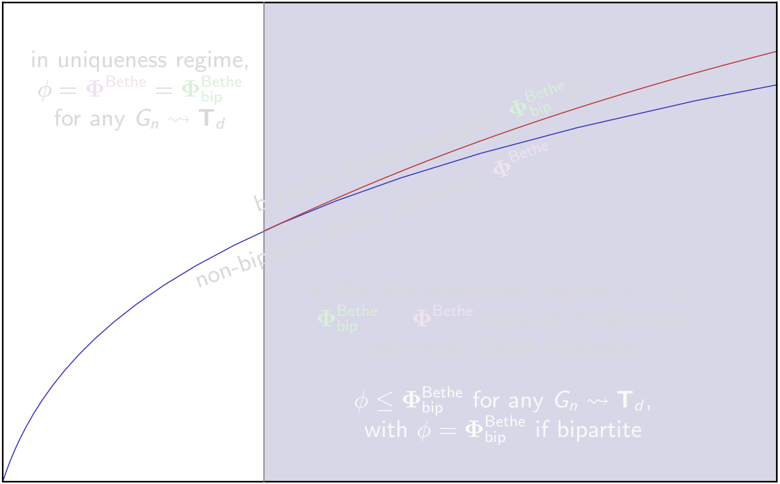
Bethe predictions and interpolation results [SS12]



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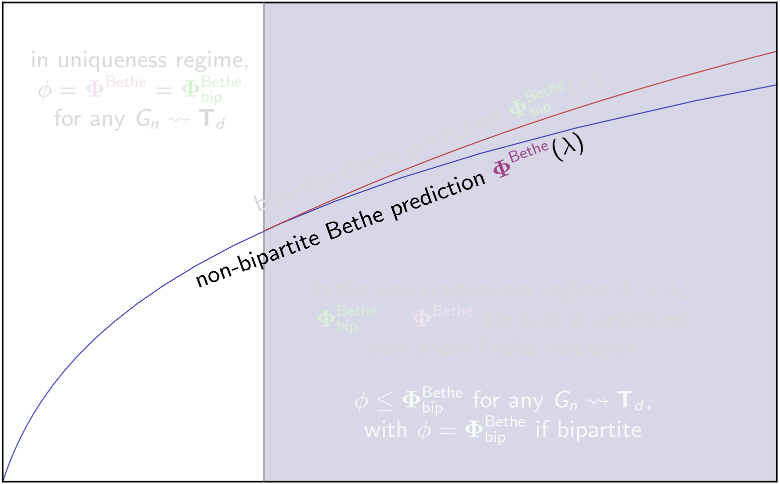

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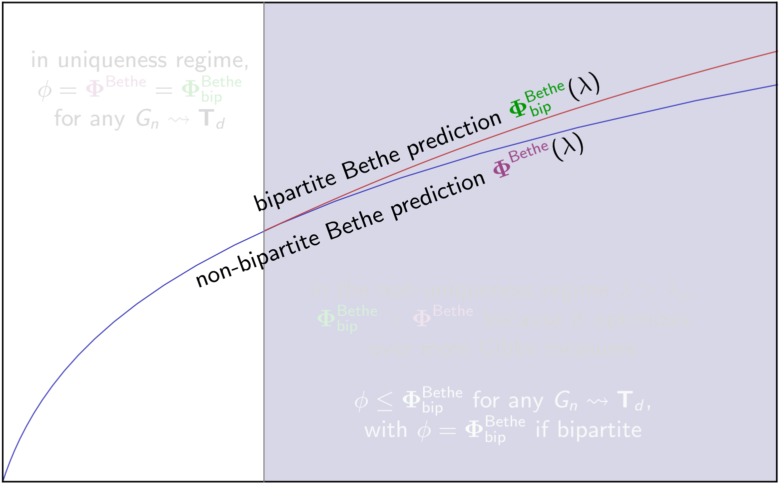

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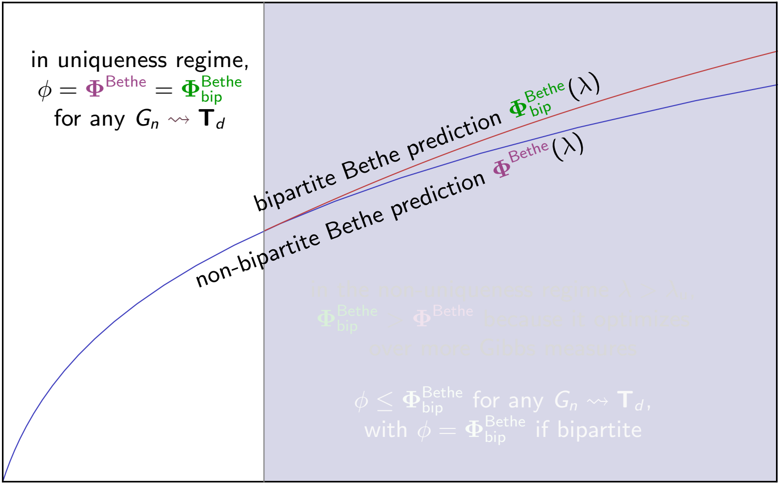

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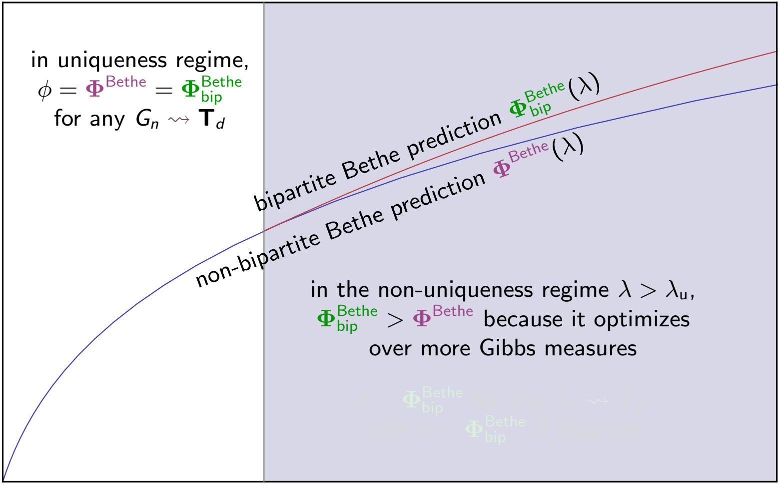

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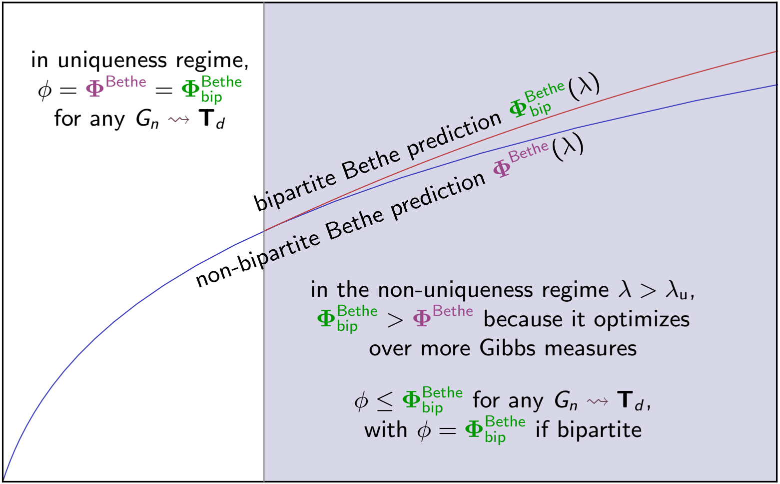

AF binary: Hardness of counting

THEOREM (Sly–Sun '12).
For hard-core (AF Ising) models in Gibbs non-uniqueness regimes, the partition function is inapproximable in polynomial time, even when restricted to regular graphs.

Combined with algorithmic results for the uniqueness regime by (ferro. Ising) Jerrum–Sinclair (1993), (hard-core) Weitz (2006), and (AF Ising) Sinclair–Srivastava–Thurley (2012), this fully characterizes the "complexity of counting" in homogeneous binary models on bounded-degree graphs

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Naturally leads to many interesting problems.

Many challenges

1. Universality: in graphs and models (e.g. Potts with general limiting tree).

2. Sharpness: Ising spin glass - push $\beta_*(0, P)$ to the RSB point.

3. Generality: Phase transition, for Ising - relax expander condition (Erdős–Rényi not included), extend to Potts.

4. Principles: Given unimodular T and pair-interaction is the free energy maximal among $G_n \rightsquigarrow T$ when G_n *bipartite*?

5. Beyond RS: Mathematical derivation of 1RSB at $\beta < \infty$ ([Ding-Sly-Sun 14] for maximal independent set; [DSS15] for k-SAT).

If you want to know more about this. . .

M. Mézard and A. Montanari, *Information, Physics, Computation*, Oxford Univ. Press. 2009.

A. Dembo and A. Montanari, *Gibbs measures and phase transitions on sparse random graphs*, Brazilian J. of Probab. and Stat. 2010.