Uniformization and percolation

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Riemann's theorem and probability

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Riemann surfaces uniformization

Poincaré (1907) proved that every simply connected Riemann surface is conformally equivalent to one of the following three surfaces: the open unit disk, the Euclidean plane, or the Riemann sphere. In particular it admits a Riemannian metric of constant curvature. This classifies Riemannian surfaces as elliptic (the shpere), parabolic (Euclidean), and hyperbolic (negatively curved).

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A discrete uniformization theorem

Every planar graph admits a circle packing Koebe (1936).

He and Schramm (1995): Let G be the 1-skeleton of a triangulation of an open disk. If the random walk on G is *recurrent*, then G is circled packed in the Euclidean plane. Conversely, if the degrees of the vertices in G are bounded and the random walk on G is *transient*, then G is circle packed in the unit disc.

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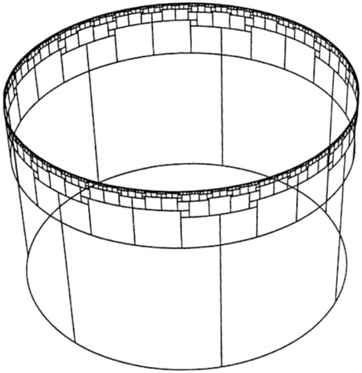
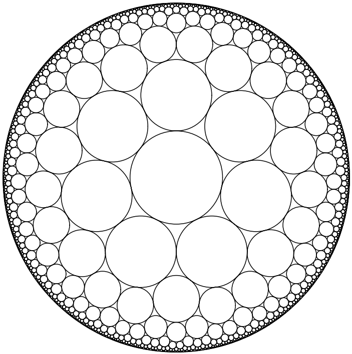
The square tiling and the circle packing of the 7-regular hyperbolic triangulation





Discrete uniformization and random walks

Using discrete uniformization with Oded (1995) we showed : A bounded degree transient planar graph admits non constant bounded harmonic functions.

Corollary: $\mathbb{Z}^3$ is not planar.

Carmesin and Georgakopoulos (2015) relaxed the condition of bounded degree in several natural cases. E.g. for non amenable planar graphs.
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The space of bounded harmonic functions

Moreover the Poisson boundary of a planar graph coincides with the boundary of its square tiling and with the boundary of its circle packing.
Recent works by Georgakopoulos and by Angel, Barlow, Gurel-Gurevich and Nachmias respectively.

Uniformization and percolation?

Let G be the 1-skeleton of a bounded degree triangulation of an open disk.

Conjecture

Assume G is transient, then $1/2$-Bernoulli site percolation on G admits an infinite cluster a.s.

Start by showing it for some fixed $p > 1/2$.

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One reason to be skeptical about the conjecture is that for critical percolation on the triangular lattice, the probability the cluster of the origin reaches distance r decays polynomially in r, while there are transient triangulations of volume growth $r^2 \log^3 r$.

Motivation for the conjecture, a short detour

Tile the unit square with (possibly infinite number) of squares of varying sizes so that at most three squares meet at corners. Color each square *black* or *white* with equal probability independently.

Conjecture

Show that there is a universal $c > 0$, so that the probability of a black *left right crossing is bigger than c. And as the size of the largest square goes to 0, the crossing probability goes to $1/2$.*

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Comments on the tiling conjecture

If true, the same should hold for a tiling, or a packing of a triangulation, with a set of shapes that are of bounded Hausdorff distance to circles.

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Comments on the tiling conjecture

Behind the tiling conjecture is a rough version of conformal invariance. That is, the crossing probability is balanced if the tiles are of uniformly bounded distance to circles (rotation invariance), and the squares can be of different sizes (dilation invariance).

From the tiling to the percolation conjecture

Let G be a 1-skeleton of bounded degree transient triangulation of an open disk. By *discrete uniformization* it admits a circle packing with similar properties as the tiling in conjecture. And if the conformal invariance heuristic holds, we will a.s. see macroscopic crossings for 1/2-Bernoulli site percolation.

Non uniqueness at $1/2$

Moreover by same reasoning we will see unboundedly many macroscopic clusters for $1/2$-Bernoulli percolation, suggesting that if G is a 1-skeleton of bounded degree transient triangulation of an open disk, then there are a.s. infinitely many infinite clusters for $1/2$-Bernoulli site percolation?

Since we believe that $p_c \leq 1/2$ for such G's we conjecture that $p_u < 1$ and uniqueness monotonicity.

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Further comments

We *believe* that $p_c \leq 1/2$ once the triangulation do not have very small (logarithmic) cut sets.

E.g. if there are $C > 0, \alpha > 0$, so that for every finite set of vertices S

$$|\partial S| > C|S|^{\alpha}.$$

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We *believe* that $p_c \geq 1/2$ for polynomial growth triangulations of the open disk. Note that if all degrees are at least 6, polynomial growth implies that vertices of higher degrees are polynomially sparse, this suggests that their critical probability for percolation is $1/2$, as of the triangular lattice. For nonamenable transitive or sofic triangulations $p_c < 1/2$, *remove the transitivity assumption?*

Further comments

Is there a closer link between the SRW and percolation, reminiscent perhaps of SLE (typically, the boundaries of both would be similar) ? Or at least a coupling, with a percolation cluster being infinite because it contains the path of a SRW (or a LERW) ?

Further comments

How does the *influence* of a square in the tiling on the crossing probability at $p = 1/2$ and it's area related?

iff?

What about a converse to the conjecture? Does recurrence imply no percolation at $1/2$?

Study similar questions in the context of magnetization.

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