Convergence of MCMC and Loopy BP in the Tree Uniqueness Region for the Hard-Core Model

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joint work with:
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Hard-Core Model

\[ G = (V, E), \ \text{fugacity} \ \lambda > 0, \ \text{for each independent set} \ \sigma \] we have

\[ \mu(\sigma) = \lambda |\sigma| / Z \]

where

\[ Z = \sum_{\sigma} \lambda |\sigma| \]

\[ Z(G, \lambda) \] is the partition function.
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$G = (V, E)$, fugacity $\lambda > 0$, for each independent set $\sigma$ we have

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where

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\[ Z = Z(G, \lambda) \] is the partition function.
The problem

For $G = (V, E)$ and fugacity $\lambda > 0$, compute the partition function

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- computationally \textit{hard problem} [Valiant 1979]
The problem

For $G = (V, E)$ and fugacity $\lambda > 0$, compute the partition function

$$Z(G, \lambda) = \sum_{\sigma} \lambda^{\left|\sigma\right|}$$

• computationally hard problem [Valiant 1979]
• focus on the approximation algorithms
Counting Vs Gibbs Marginals
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Compute $Z(G, \lambda)$
Counting Vs Gibbs Marginals

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for $\sigma$ distributed as in $\mu$ compute

$$\Pr[\sigma = \emptyset] = \frac{1}{Z(G, \lambda)}$$
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$$\Pr[\sigma = \emptyset] = \frac{1}{Z(G, \lambda)}$$

$$\Pr[\sigma = \emptyset] = \Pr \left[ \bigcap_i \sigma(u_i) = \text{unoccupied} \right]$$
Counting Vs Gibbs Marginals

Compute $Z(G, \lambda)$

for $\sigma$ distributed as in $\mu$ compute

$$\Pr[\sigma = \emptyset] = \frac{1}{Z(G, \lambda)}$$

$$\Pr[\sigma = \emptyset] = \prod_i \Pr \left[ \sigma(u_i) = \text{unoccupied} \mid \bigcap_{j < i} \sigma(u_j) = \text{unoccupied} \right]$$
Belief Propagation - An exact algorithm for trees
Belief Propagation - An exact algorithm for trees

For $T$ and $\lambda$

\[
q(v, w) = \mu(v \text{ occupied} | w \text{ unoccupied})
\]

\[
R_{v \rightarrow w} = q(w(v))
\]

\[
R_{v \rightarrow p(v)} = \lambda \prod_{w \in N(v)} \{p(v)\}
\]

For every $i \geq 1$

\[
R_{i v \rightarrow p(v)} = \lambda \prod_{w \in N(v)} \{p(v)\}
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Belief Propagation - An exact algorithm for trees

For $T$ and $\lambda$

For every $i \geq 1$

$$R_i v \rightarrow p(v) = \lambda \prod_{w \in N(v)} \{ p(v) \} \frac{1}{1 + R_{i-1} w \rightarrow v}$$
Belief Propagation - An exact algorithm for trees

For $T$ and $\lambda$

$$q(v, w) = \mu(v \text{ occupied}|w \text{ unoccupied})$$
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For \( T \) and \( \lambda \)

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q(v, w) = \mu(v \text{ occupied}|w \text{ unoccupied})
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R_{v \rightarrow w} = \frac{q(v, w)}{1 - q(v, w)}
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R_v = \lambda \prod_{z \in N(v)\setminus\{w\}} \frac{1}{1 + R_{z \rightarrow v}}
\]

Start from arbitrary $R^0_{v \rightarrow w} s$, iterate like

\[
R^i_{v \rightarrow w} = \lambda \prod_{z \in N(v)\setminus\{w\}} \frac{1}{1 + R^{i-1}_{z \rightarrow v}}
\]
BP and Gibbs distribution on trees
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Convergence on trees

There exists $i_0$ such that for every $i \geq i_0$ and every $(R^0_{v \rightarrow w})_{\{v,w\} \in E}$ we have

$$R^i_{v \rightarrow w} = R^*_{v \rightarrow w}$$

In turn

$$\mu(v \text{ occupied}| w \text{ unoccupied}) = q^* = \frac{R^*_{v \rightarrow w}}{1 + R^*_{v \rightarrow w}}$$
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BP is an elaborate version of Dynamic Programing
Algorithmic Approaches
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Compute *estimates of Gibbs marginals*
Algorithmic Approaches

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- Deterministic
Algorithmic Approaches

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  - *Numerical* estimations of Gibbs marginals

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  - Estimation by using *samples (approximately) Gibbs distributed*
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Compute estimates of Gibbs marginals

- Deterministic
  - Numerical estimations of Gibbs marginals
  - Fully Polynomial Time Approximation Scheme (FPTAS)

- Randomized
  - Estimation by using samples \((approximately)\) Gibbs distributed
  - Fully Polynomial Time Randomized Approximation Scheme (FPRAS)
Algorithmic Approaches

Compute *estimates of Gibbs marginals*

- **Deterministic**
  - *Numerical* estimations of Gibbs marginals
  - Fully Polynomial Time Approximation Scheme (FPTAS)
    - in time $\text{poly}(n, \epsilon^{-1})$
    \[ \hat{Z} \in (1 \pm \epsilon)Z(G, \lambda) \]

- **Randomized**
  - Estimation by using *samples (approximately) Gibbs distributed*
  - Fully Polynomial Time Randomized Approximation Scheme (FPRAS)
    - in time $\text{poly}(n, \epsilon^{-1}, \log(\delta^{-1}))$
    \[ \Pr[\hat{Z} \in (1 \pm \epsilon)Z(G, \lambda)] \geq 1 - \delta \]
For which $\lambda$ can we approximate?
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Intuition

... the larger $\lambda$ the harder is to approximate $Z(G, \lambda)$
For which $\lambda$ can we approximate?

Hardness of approximation [Sly 2010]

For triangle-free $\Delta$-regular graphs, where $\Delta \geq 3$, and for all $\lambda > \lambda_c(\Delta)$, it is NP-hard to approximate the partition function within factor $e^{\Theta(n)}$. 
For which $\lambda$ can we approximate?

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- Galanis, Ge, Stefankovic, Vigoda, Yang (2011)
- Sly, Sun (2012)
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**Hardness of approximation [Sly 2010]**

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**What is $\lambda_c(\Delta)$? [Kelly 1985]**

Critical point for “uniqueness/non-uniqueness” transition of the hard-core model on $\Delta$ regular trees

$$
\lambda_c(\Delta) := \frac{(\Delta - 1)^{\Delta - 1}}{(\Delta - 2)^\Delta} \sim \frac{e}{\Delta}
$$
Tree Uniqueness
For $\Delta$-regular tree of height $\ell$:

Let $p_\ell = \mu$ (root is occupied)

Extremal cases: even versus odd height.
Tree Uniqueness

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Extremal cases: even versus odd height.

Does \( \lim_{\ell \to \infty} p_{2\ell} = \lim_{\ell \to \infty} p_{2\ell+1} \)?
For $\Delta$-regular tree of height $\ell$:

Let $p_\ell = \mu$ (root is occupied)

Extremal cases: even versus odd height. Does $\lim_{\ell \to \infty} p_{2\ell} = \lim_{\ell \to \infty} p_{2\ell+1}$?

$\lambda \leq \lambda_c(\Delta)$: No boundary effects root.
$\lambda > \lambda_c(\Delta)$: Exist boundaries effect root.
Deterministic Algorithms

Weitz's approach [Weitz 2006]

- Given $G$ of maximum degree $\Delta$ and $\lambda < \lambda_c(\Delta)$,
  - uses tree of self avoiding walks, to organize the computations
  - reduces to dynamic programming.
  - the size of the computation depends on the size of the tree
  - in the worst case the tree is exponentially large
  - "prune" the tree and still be accurate
    - this step requires $\lambda < \lambda_c$
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Approximation guarantees
For all $\delta > 0$, there exists constant $C = C(\delta) > 0$, for all $\Delta$ all $G$ of maximum degree $\Delta$, all $\lambda < (1 - \delta)\lambda_c(\Delta)$ all $\epsilon > 0$ Weitz’s algorithm returns an estimation $\hat{Z}$ of the partition function $Z(G, \lambda)$ such that

$$(1 - \epsilon)Z(G, \lambda) \leq \hat{Z} \leq (1 + \epsilon)Z(G, \lambda)$$

in time $O((n/\epsilon)^{C \log \Delta})$. 

Performance Weitz’s algorithm

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- Li, Lu, and Yin (2012), (2013)
- Sinclair, Srivastava and Yin (2013)
Randomized Algorithm

Given $G$ and $\lambda > 0$,
- set up an ergodic Markov Chain over the independent sets
- the equilibrium distribution is the hard-core model with fugacity $\lambda$
- the algorithm simulates the Markov chain
- outputs the configuration of the chain after "sufficiently many" steps
  the output should be close to the equilibrium distribution
  it is desirable that the chain mixes "fast"
Randomized Algorithm

Markov Chain Monte Carlo

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Glauber dynamics ($X_t$)

1. Choose $v$ uniformly at random from $V$.

   $$X' = \begin{cases} 
   X_t \cup \{v\} & \text{with probability } \frac{\lambda}{1 + \lambda} \\
   X_t \setminus \{v\} & \text{with probability } \frac{1}{1 + \lambda}
   \end{cases}$$

2. If $X'$ is an independent set, then $X_{t+1} = X'$; otherwise, $X_{t+1} = X_t$.

The chain converges to the hard-core model with fugacity $\lambda$. 
Glauber dynamics \( (X_t) \)

\( X_t \rightarrow X_{t+1} \) is defined as follows:

1. Choose \( v \) uniformly at random from \( V \).
   - \( X'_t = X_t \cup \{v\} \) with probability \( \lambda / (1 + \lambda) \).
   - \( X'_t = \{v\} \) with probability \( 1 / (1 + \lambda) \).
2. If \( X'_t \) is independent set, then \( X_{t+1} = X'_t \), otherwise \( X_{t+1} = X_t \).

The chain converges to the hard-core model with fugacity \( \lambda \).
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Our Results

Theorem

For all $\delta > 0$, there exists $\Delta_0 = \Delta_0(\delta)$ for all graphs $G = (V, E)$ of maximum degree $\Delta \geq \Delta_0$ and girth $\geq 7$, all $\lambda < (1 - \delta)\lambda^c(\Delta)$, the mixing time of the Glauber dynamics satisfies $T_{\text{mix}} = O(n \log(n))$. 
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Mixing Time . . .

$$T_{mix} = \min\{t : \text{ for all } X_0, d_{tv}(X_t, \mu) \leq 1/4\},$$
Our Results

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For all $\delta > 0$, there exists $\Delta_0 = \Delta_0(\delta)$ for all graphs $G = (V, E)$ of maximum degree $\Delta \geq \Delta_0$ and girth $\geq 7$, all $\lambda < (1 - \delta)\lambda_c(\Delta)$, the mixing time of the Glauber dynamics satisfies

$$T_{\text{mix}} = O\left(n \log(n)\right).$$

Corollary
An $O^*(n^2)$ FPRAS for estimating the partition function $Z$. 
Our Results

Theorem
For all $\delta > 0$, there exists $\Delta_0 = \Delta_0(\delta)$ for all graphs $G = (V, E)$ of maximum degree $\Delta \geq \Delta_0$ and girth $\geq 7$, all $\lambda < (1 - \delta)\lambda_c(\Delta)$, the mixing time of the Glauber dynamics satisfies

$$T_{mix} = O(n \log(n)).$$

Previous work
$T_{mix} = O(n \log(n))$ for Glauber dynamics on $G$ of maximum degree $\Delta$ and $\lambda < 2/((\Delta - 2)$

- Dyer Greenhill, Luby, Vigoda
$O(n \log n)$ mixing for Random Graphs

Corollary

$T_{\text{mix}} = O(n \log n)$ for Glauber dynamics with $\lambda \leq (1 - \delta)\lambda_c(\Delta)$ for

- random $\Delta$-regular graph
- random $\Delta$-regular bipartite graph

Mossel, Weitz, Wormald (2009)
Relaxation for girth

“ # short cycles in the neighborhood of each vertex in $G$ are not too many”
**O(n log n)** mixing for Random Graphs

**Relaxation for girth**

“# short cycles in the neighborhood of each vertex in G are not too many”

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\[ T_{mix} = O(n \log n) \] for Glauber dynamics with \( \lambda \leq (1 - \delta)\lambda_c(\Delta) \) for

\begin{itemize}
  \item random \( \Delta \)-regular graph
  \item random \( \Delta \)-regular bipartite graph
\end{itemize}

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Belief Propagation on trees

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$\mu(v \text{ occupied} | w \text{ unoccupied})$

$q(v, w) = \mu(v \text{ occupied} | w \text{ unoccupied})$

\[ R_{v \rightarrow w} = \frac{q(v, w)}{1 - q(v, w)} \]

\[ R_{v \rightarrow w} = \lambda \prod_{z \in N(v) \setminus \{w\}} \frac{1}{1 + R_{z \rightarrow v}} \]

BP starts from arbitrary $R_{v \rightarrow w}^0$s, iterates like

\[ R_{v \rightarrow w}^i = \lambda \prod_{z \in N(v) \setminus \{w\}} \frac{1}{1 + R_{z \rightarrow v}^{i-1}} \]
(Loopy) Belief Propagation
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Nothing prevents to use Belief propagation for graph with cycles.
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Nothing prevents to use Belief propagation for graph with cycles.
  • We do not know whether it converges
(Loopy) Belief Propagation

Nothing prevents to use Belief propagation for graph with cycles.

- We do not know whether it converges
- \ldots if does, we do not know where exactly it converges
BP Convergence for girth $\geq 6$
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\[ R^i_{v \rightarrow w} = \lambda \prod_{z \in N(v) \setminus \{w\}} \frac{1}{1 + R^{i-1}_{z \rightarrow v}} \]
BP Convergence for girth $\geq 6$

\[ R^i_{v\rightarrow w} = \lambda \prod_{z \in N(v) \setminus \{w\}} \frac{1}{1 + R^i_{z\rightarrow v}} \quad \text{and} \quad q^i(v, w) = \frac{R^i_{v\rightarrow w}}{1 + R^i_{v\rightarrow w}} \]
BP Convergence for girth $\geq 6$

$$R^i_{v \rightarrow w} = \lambda \prod_{z \in N(v) \setminus \{w\}} \frac{1}{1 + R^i_{z \rightarrow v}}$$

and

$$q^i(v, w) = \frac{R^i_{v \rightarrow w}}{1 + R^i_{v \rightarrow w}}$$

**Theorem**

Let $\delta, \epsilon > 0$, $\Delta_0 = \Delta_0(\delta, \epsilon)$ and $C = C(\delta, \epsilon)$. For $G$ of max degree $\Delta \geq \Delta_0$ and girth $\geq 6$, all $\lambda < (1 - \delta)\lambda_c(\Delta)$, the following holds: for $i \geq C$, for all $v \in V$, $w \in N(v)$,

$$\left| \frac{q^i(v, w)}{\mu(v \text{ is occupied} \mid w \text{ is unoccupied})} - 1 \right| \leq \epsilon$$
BP Convergence for girth $\geq 6$

\[ R_{v \rightarrow w}^i = \lambda \prod_{z \in N(v) \setminus \{w\}} \frac{1}{1 + R_{z \rightarrow v}^{i-1}} \quad \text{and} \quad q^i(v, w) = \frac{R_{v \rightarrow w}^i}{1 + R_{v \rightarrow w}^i} \]

**Theorem**

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\[ \left| \frac{q^i(v, w)}{\mu(\text{v is occupied} \mid \text{w is unoccupied})} - 1 \right| \leq \epsilon \]

we also have convergence for the BP estimate of $\mu(\text{v is occupied})$
Path Coupling for bounding $T_{\text{mix}}$
Path Coupling for bounding $T_{mix}$

Path Coupling [Bubley and Dyer 1997]
Path Coupling for bounding $T_{mix}$

Path Coupling [Bubley and Dyer 1997]
Consider copies $(X_s), (Y_s)$ such that $X_t \oplus Y_t = \{v\}$
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\Phi(X, Y) = \sum_{u \in X \oplus Y} \Phi_u
$$
Path Coupling Example
Path Coupling Example
Path Coupling Example

Expected distance

\[ E[\Phi(X_{t+1}, Y_{t+1})| X_t, Y_t] = \left(1 - \frac{1}{n}\right) \Phi_v + \sum_{z_i} \Pr[z_i \in Y_{t+1}] \cdot \Phi_{z_i} \]
Path Coupling Example

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Path Coupling Example

Expected distance

$$\mathbb{E} [ \Phi(X_{t+1}, Y_{t+1}) | X_t, Y_t] = \left( 1 - \frac{1}{n} \right) \Phi_v + \frac{1}{n} \sum_{z_i} 1\{z_i \text{ unblocked}\} \frac{\lambda}{1 + \lambda} \Phi_z$$
Path Coupling Example

Path coupling condition

\[ \Phi_v > \frac{\lambda}{1 + \lambda} \sum_{z_i} 1\{z_i \text{ unblocked in } Y_t\} \cdot \Phi_{z_i} \]
Key Results

- We don’t know a $\Phi$ that gives contraction for worst-case $X_t, Y_t$.
- We can find $\Phi$ when locally $X_t, Y_t$ “behave” like $\omega^*$.
- Glauber dynamics converges locally to $\omega^*$
  - dynamics gets *local uniformity*
- Given $\Phi$ and convergence of Glauber dynamics we show rapid mixing
Unblocked Neighbors and loopy BP

\[
\omega_i(z) = \prod_{y \sim z} \omega_{i-1}(y) + \lambda \cdot \omega_i(z)
\]

is the loopy BP estimate of \(z\) to be unblocked.

\(\omega^* \approx \mu\) (\(z\) is unblocked)
Unblocked Neighbors and loopy BP

\[ \omega_i^z = \prod_{y \sim z} \frac{1}{1 + \lambda \cdot \omega_y^{i-1}} \]
Unblocked Neighbors and loopy BP

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Back to Path Coupling
Back to Path Coupling

\[ \Phi \nu > \lambda + \lambda \sum z_i \{ z_i \text{ unblocked} \} \cdot \Phi z_i \]

when \( X_t, Y_t \) "behave" like \( \omega^* \)

\[ \Phi \nu > \lambda \sum z_i \omega^*(z_i) \cdot \Phi z_i \]
Back to Path Coupling

worst case condition

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Finding $\Phi$

There is $\Phi$ such that

\[(1 - \delta) \Phi v \geq \sum z_i \lambda \omega^* (z_i) 1 + \lambda \omega^* (z_i) \Phi z_i\]

$\times n$ matrix $C(v, z)$ =

\[
\begin{cases}
\lambda \omega^* (z_i) 1 + \lambda \omega^* (z_i) & \text{if } z_i \in N(v) \\
0 & \text{otherwise}
\end{cases}
\]

There is a vector $\Phi \in \mathbb{R}^V > 0$ such that $C \Phi \leq (1 - \delta) \Phi$. 
Finding $\Phi$

Reformulation

There is $\Phi$ such that

$$(1 - \delta) \Phi v \geq \sum z_i \lambda \omega^* (z_i) + \lambda \omega^* (z_i) \Phi z_i$$

$n \times n$ matrix $C(v, z) =$

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$n \times n$ matrix $C$

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Connections with Loopy BP

\[ \omega_i(z) = \prod_{y \sim z} 1 + \lambda \cdot \omega_i - 1 y \]

\[ J^* = J |_{\omega = \omega^*} \]

denote the Jacobian of BP at the fixed point \( \omega^* \).

Relation to Path Coupling

\[ C = D - 1 J^* D, \]

where \( D \) is diagonal matrix, with \( D(v, v) = \omega^*(v) \).
Connections with Loopy BP

Jacobian of Loopy BP

\[ \omega_i \propto \prod_{y \sim z} 1 + \lambda \cdot \omega_{i-1} \]

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\[ C = D^{-1} J^* D, \]

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Covvergence from loopy BP

There is a vector $\Phi \in \mathbb{R}^V > 0$ such that

$$ (D - 1 J^* D) \Phi \leq (1 - \delta) \cdot \Phi $$

has the same eigenvalues as $J^*$. Spectral radius of BP in uniqueness region

We should expect $\rho(J^*) < 1$, because the fixed point $\omega^*$ is attractive.
Covergence from loopy BP

Reduction to BP Spectral radius

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Local Uniformity

Theorem
Let $\epsilon, \delta > 0$, $\Delta_0 = \Delta_0(\epsilon, \delta)$ and $C = C(\epsilon, \delta)$. Let $G$ of max degree $\Delta$, for $\Delta > \Delta_0$, and girth $\geq 7$. For $(X_t)$ the Glauber dynamics with fugacity $\lambda < (1 - \delta)\lambda_c(\Delta)$ and any $v$ the following holds: With probability $1 - \exp(-\Delta/C)$, we have that

$$\# \text{ Unblocked Neighbors of } v \text{ in } X_t < \sum_{z \in N(v)} \omega^*(z) + \epsilon \Delta$$

where $t \geq Cn \log \Delta$. 
Key Results

• We don’t know a $\Phi$ that gives contraction for worst-case $X_t, Y_t$.
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Rapid Mixing with uniformity
Dyer, Frieze, Hayes, Vigoda 2013
There is a single disagreement at $\nu$
Rapid Mixing with uniformity
Dyer, Frieze, Hayes, Vigoda 2013

Run the chains for $Cn \log \Delta$ steps, "burn-in"
Rapid Mixing with uniformity
Dyer, Frieze, Hayes, Vigoda 2013

Run the chains for $Cn \log \Delta$ steps, “burn-in”
The disagreements spread in the graph during burn-in
Rapid Mixing with uniformity

Dyer, Frieze, Hayes, Vigoda 2013

Typically the disagreements do not escape the ball
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Typically the ball has uniformity.
Interpolate and do path coupling for the pairs, ... pairs with have local uniformity
Interpolate and do path coupling for the pairs, 
... pairs with have local uniformity and $\Phi$ gives contraction
Rapid Mixing with uniformity
Dyer, Frieze, Hayes, Vigoda 2013

\[ E \left[ \Phi(X_{C' n \log \Delta}, Y_{C' n \log \Delta}) \mid X_0, Y_0 \right] \leq (1 - \gamma) \Phi(X_0, Y_0) \]
Key Results

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Local uniformity I

\[ R(\sigma, v) = \prod_{w \sim v} \left( 1 - \frac{\lambda}{1+\lambda} 1\{w \text{ unblocked by its children}\} \right), \]
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\[ R(\sigma, v) = \Pr_{Y \sim \mu} [v \text{ is unblocked in } Y | v \not\in Y, Y(S_2(v)) = \sigma(S_2(v))] \]
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BP for Gibbs measure

Let \( \gamma, \delta > 0 \), \( \Delta_0 = \Delta_0(\gamma, \delta) \) and \( C = C(\gamma, \delta) \). Let \( G \) be of girth \( \geq 6 \) and maximum degree \( \Delta > \Delta_0 \). Let \( X \) be distributed as in \( \mu \) with \( \lambda < (1 - \delta) \lambda_c(\Delta) \).

Then for any vertex \( v \) with probability \( \geq 1 - \exp(-\Delta/C) \) it holds that

\[
\left| R(X, v) - \prod_{z \sim v} \left( 1 - \frac{\lambda}{1 + \lambda} R(X, z) \right) \right| < \gamma.
\]
Local uniformity

$$R(\sigma, v) = \prod_{w \sim v} \left( 1 - \frac{\lambda}{1 + \lambda} \right) \mathbf{1}\{w \text{ unblocked by its children}\},$$

BP for Glauber dynamics

Let $G$ be of girth $\geq 7$ and maximum degree $\Delta > \Delta_0$. Let $(X_t)$ be the Glauber dynamics with $\lambda < (1 - \delta)\lambda_c(\Delta)$. Then for any vertex $v$ and any $t > Cn \log \Delta$ with probability $\geq 1 - \exp(-\Delta/C)$ it holds that

$$\left| R(X_t, v) - \prod_{z \sim v} \left( 1 - \frac{\lambda}{1 + \lambda} \mathbb{E}_{t_z} [R(X_{t_z}, z)] \right) \right| < \gamma.$$
Let $G$ be of girth $\geq 7$ and maximum degree $\Delta > \Delta_0$. Let $(X_t)$ be the Glauber dynamics with $\lambda < (1 - \delta) \lambda_c(\Delta)$. For all $I = [t_0, t_1]$, where $t_0 = Cn \log \Delta$, for every $v \in V$ with probability $1 - (1 + |I|/n) \exp (-\Delta/C)$, we have that $|R(X_t, v) - \omega^* (v)| \leq \epsilon$. 
Lemma
Let $G$ be of girth $\geq 7$ and maximum degree $\Delta > \Delta_0$. Let $(X_t)$ be the Glauber dynamics with $\lambda < (1 - \delta) \lambda_c(\Delta)$.
For all $\mathcal{I} = [t_0, t_1]$, where $t_0 = Cn \log \Delta$, for every $v \in V$ with probability $1 - (1 + |\mathcal{I}|/n) \exp(-\Delta/C)$, we have that

$$(\forall t \in \mathcal{I}) \quad |R(X_t, v) - \omega^*(v)| \leq \epsilon.$$
Iterations in space and time
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Convergence with $\Psi$

Potential function

$$\Psi(x) = (\lambda)^{-1} \text{arcsinh}(\sqrt{\lambda}x)$$
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Provided

- $t \in I'$ approximate BP equation hold in $B(v, R)$ \(\forall t \in I_{i+1},\)
  
  $$u \in B(v, i + 1)$$

  $$|\Psi(R(X_t, u)) - \Psi(\omega^*(u))| \leq \alpha_{i+1}$$

\(\forall t \in I_i, u \in B(v, i)\)

$$|\Psi(R(X_t, u)) - \Psi(\omega^*(u))| \leq (1 - \delta)\alpha_{i+1}$$
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Concluding Remarks

• Rapid mixing for Glauber Dynamics
• $\Delta > \Delta_0$ and girth $\geq 7$
• $\lambda$ in uniqueness
• Path coupling with local uniformity
• "Hamming weights"
• Novel connection between Path Coupling and Loopy BP for both uniformity and Hamming weights
• Techniques from Glauber dynamics to analyze Loopy BP for graphs of girth $\geq 6$ in the uniqueness region
• The connection between Glauber dynamics and Loopy BP is deep
• Allows to establish uniformity and weights in a systematic way
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  - local uniformity
  - “Hamming weights”
- novel connection between Path Coupling and Loopy BP
  - for both uniformity and Hamming weights
- Techniques from Glauber dynamics to analyze Loopy BP
  - for graphs of girth $\geq 6$ in the uniqueness region
- The connection between Glauber dynamics and Loopy BP is deep
  - Allows to establish uniformity and weights in a systematic way
The End

THANK YOU!