Studying cutoff in card shuffling

Evita Nestoridi

Princeton University

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Random to random

The model:

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The mathematical setup

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Many steps of the walk

$P^t(id, g)$ gives the probability of moving from the identity to g in t steps.

The Question

Convergence

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Question: How many steps until the deck is shuffled well enough?

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Mixing time:

$$t_{mix}(\epsilon) = \min\{t \in \mathbb{N} : ||P_{id}^t - U||_{T.V.} \leq \epsilon\}$$

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5. Saloff-Coste and Zuniga (2008) imporved the upper bound to $2 n \log n$, Morris and Qin (2014) improved to $1.5324 n \log n$.

Result

Theorem (M.Bernstein, E.N.)

Let $= \frac{3}{4} n \log n + cn$, then

$$||P_{id}^{*t} - U||_{T.V} \leq e^{-c}$$

where $c > 0$.

Result

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Theorem (M.Bernstein, E.N.)

$$\lim_{c\to\infty}\lim_{n\to\infty}||P_{id}^{*(\frac{3}{4}n\log n+cn)}-U||_{T.V.}=0$$

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Theorem (Subag)

$$\lim_{c\to\infty} \lim_{n\to\infty} ||P_{id}^{*(\frac{3}{4} n \log n - cn)} - U||_{T.V} = 1$$

Cutoff

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The sequence of walks on a space $\mathcal{A}_n$ exhibits cutoff at t_n with window $w_n = o(t_n)$ if and only if

$$\lim_{c \to \infty} \lim_{n \to \infty} d(t_n - cw_n) = 1 \text{ and } \lim_{c \to \infty} \lim_{n \to \infty} d(t_n + cw_n) = 0.$$

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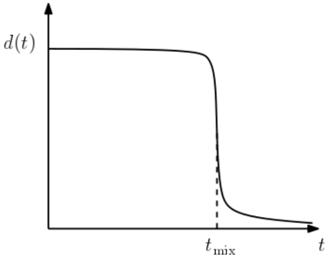

Figure: Cutoff diagram.

Upper Bound Theorem

- Our matrix is

$$P(x, xg) = \begin{cases} \frac{1}{n^2}, & \text{if } g = (a, a+1, \ldots, b)^{\pm 1} \in S_n \text{ with } b - a > 1 \\ \frac{2}{n^2}, & \text{if } g = (a, a+1, \ldots, b)^{\pm 1} \in S_n \text{ with } b - a = 1 \\ \frac{1}{n}, & \text{if } g = id \\ 0, & \text{otherwise.} \end{cases}$$

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Let $\lambda_j, j \in \{0, 1, 2, ..., |G| - 1\}$ be the eigenvalues of P. Then:

$$4||P_{id}^{*t} - U||_{T.V.}^2 \leq \sum_{j=1}^{|G|-1} \lambda_j^{2t}$$

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$$\lambda_k = 1 - \frac{n + k^2 + k}{n^2}.$$

for $k = 0, \dots n - 2$.

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An important case

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A conjecture turns out to be wrong

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The biggest eigenvalue term is

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and it gives that

$$t=\frac{1}{2}n\log n+cn$$

steps are sufficient to make it small.

The Bernoulli-Laplace model

Shuffling large decks of cards

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Casinos, Cryptography and games

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- It is suggested to shuffle like this in board games, for k close to $\frac{n}{2}$.

An equivalent process

The Bernoulli-Laplace urn model

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- $P^t(i,j)$ gives the probability of moving from i to j in t steps.
- Each row of P converges to

$$\pi_n(j) = \frac{\binom{n}{j}\binom{n}{n-j}}{\binom{2n}{n}} \quad 0 \leq j \leq n.$$

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- Diaconis and Pal (2017) studied shuffling by shmooshing, which is a technique used by casinos.

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Theorem(A.Eskenazis, E.N)

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The first two eigenvectors and eigenvalues

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The second e-value and e-vector

The second eigenvalue is $1 - \frac{2k(k-1)(n-k)}{n^2(n-1)}$ and the corresponding eigenvector is $f_2(x) = 1 - \frac{2x(x-1)(n-x)}{n^2(n-1)}$.

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2. $X_0 = 0$.
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Total variation distance

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Coupling inequality

Let T be the first time that $X_t = Y_t$. Then

$$||P_{id}^t - \pi_n||_{T.V.} \leq \mathbb{P}(T > t)$$

The coupling

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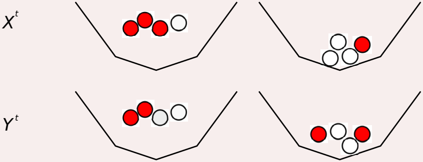

The coupling

On each pair, enumerate the balls of the left urn with the numbers $\{1, \dots n\}$, starting with the red ones.

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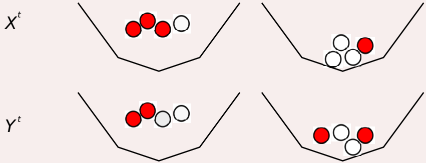

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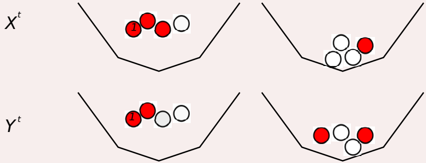

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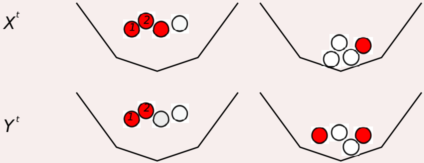

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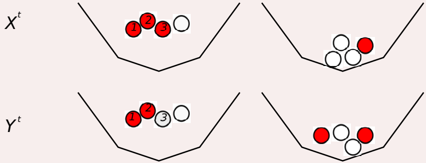

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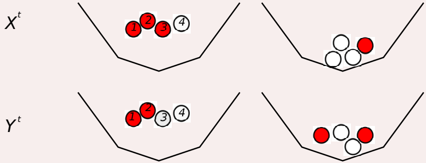

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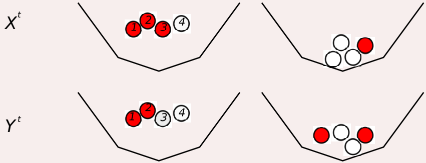

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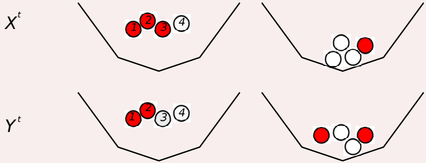

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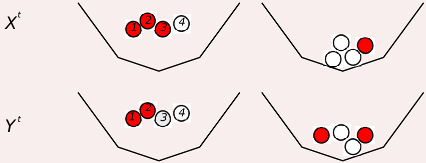

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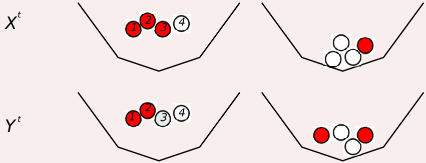

Enumerate the balls of the right urns with $\{n+1, \ldots 2n\}$, starting with the red ones. Pick k numbers from $\{1, \ldots n\}$ without replacement and k numbers from $\{n+1, \ldots 2n\}$. On each pair, switch the balls with the corresponding numbers.

The coupling is not optimal

The issue

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The issue

We have that

$$\mathbb{E}(|X_t - Y_t||X_0, Y_0) \leq n\left(1 - \frac{2k(n-k)}{n^2}\right)^t$$

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We have that

$$\mathbb{E}(|X_t - Y_t||X_0, Y_0) \leq n \left(1 - \frac{2k(n-k)}{n^2}\right)^t$$

and this gives that

$$t_{mix}(e^{-c}) \leq \frac{n^2}{2k(n-k)} \log n + c\frac{n^2}{2k(n-k)}$$

A more complicated coupling

Bringing X_t and Y_t within distance $\sqrt{n}$ is easy.

We start with the marking scheme until $t = \frac{n}{4k} \log n$.

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Lemma

Let $t \geq \frac{n}{4k} \log n$. Then

$$\left| \mathbb{E}(X_t | X_0) - \frac{n}{2} \right| \leq \sqrt{n} \quad \text{and} \quad \mathrm{Var}\left(X_t | X_0\right) \leq n. \tag{2}$$

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X_t, Y_t remain close for a while

Doob's maximal inequality gives that w.h.p. for the next $d\frac{n}{k}$ steps, we will have $|X_t - Y_t| \leq \sqrt{n}$.

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Hitting time lemma

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Keep on running the two chains independently. Let $\tau = \min\{t : |X_t - Y_t| \leq \sqrt{k}\}$.

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Keep on running the two chains independently. Let $\tau = \min\{t : |X_t - Y_t| \leq \sqrt{k}\}$. Then

$$\mathbb{P}\left(\tau > d\frac{n}{k}\right) \leq \frac{1}{\sqrt{d}}$$

A more complicated coupling

The final step

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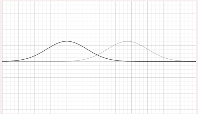

Figure: At distance $\sqrt{k}$

The final step

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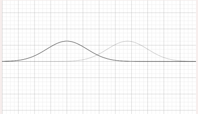

Figure: At distance $\sqrt{k}$

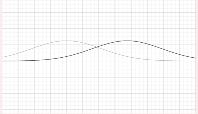

Figure: After three steps.

The result

Theorem(A.Eskenazis, E.N)

If $k = o(n^{2/3})$ and $t_{n,c} = \frac{n}{4k} \log n + c\frac{n}{k}$, then

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If $k = o(n^{2/3})$ and $t_{n,c} = \frac{n}{4k}\log n + c\frac{n}{k}$, then

$$\lim_{c\to\infty}\lim_{n\to\infty} ||P_{id}^{t_{n,c}} - \pi_n||_{T.V.} = 0$$

Open Questions

1. What about more general k?

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2. What about multiple urns?

Thank You!